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Construction, Completeness Proof and Empirical Study of Cross-Border E-Commerce Market Measure Space

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Abstract: Against the backdrop of the deep integration of the global digital economy and cross-border e-commerce, and addressing the lack of measure theory and quantitative analysis challenges for high-dimensional dynamic data in this field, this study constructs a measure theory system for cross-border e-commerce markets that combines mathematical rigor and economic interpretability based on Carathéodory's extension theorem. Based on functional analysis and measure theory, the study defines the market fundamental set as the topological product space of a time index set and a multi-dimensional transaction state space. By constructing a combined structure of a left-open right-closed interval semiring and a power set semiring that satisfies the closure of Boolean algebra operations, an algebraic framework is established for the unified measurement of continuous and discrete variables. On the semiring structure, a σ-finite premeasure integrating Lebesgue measure and counting measure is defined. With the help of the countable covering mechanism generated by outer measure and the measure screening rules of Carathéodory's measurability condition, the axiomatic extension from premeasure to complete measure on the σ-algebra is completed. Through the verification of Carathéodory's condition for subsets of null sets and the transmission of outer measure monotonicity, the completeness of the measure space is strictly proved, and the core property that "subsets of null sets must be measurable" is established, providing a solid measure-theoretic foundation for mathematical modeling of cross-border e-commerce markets. At the empirical analysis level, the study uses micro-panel data on global cross-border e-commerce transactions from 2018 to 2024. Through the Kolmogorov-Smirnov test in non-parametric hypothesis testing, the distribution isomorphism between the theoretical measure and empirical data is verified. Based on the measure space theory, a Generalized Method of Moments (GMM) panel regression model is constructed. System GMM and Difference GMM estimation methods are used to handle endogeneity issues. Combined with instrumental variable methods and lag variable techniques, key parameters such as the logarithmic elasticity of economic scale between importing and exporting countries, the spatial decay effect of geographical distance, and the asymmetric inhibitory effect of tariff policies are quantitatively analyzed. A graph neural network model integrating measure theory is innovatively designed. By introducing a completeness regular term, the measure constraints on null sets and their subsets are achieved. Combined with the SHAP value interpretability analysis method, the marginal contribution of each characteristic variable in model decision-making is revealed. The study finds that the constructed measure space not only satisfies the axiomatic requirements of modern measure theory such as completeness and σ-finiteness, but also through the empirical tests of the GMM model and graph neural network, it is confirmed that it can effectively characterize the economic scale effect, spatial distance decay law, and policy sensitivity characteristics in cross-border e-commerce transactions, providing a methodological innovation paradigm based on measure theory for quantitative analysis in the field of international business in the digital economy era.

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1. Introduction

1.1. Research Background and Problem Formulation

Driven by the deep integration of the global digital economy and the restructuring of the international division of labor, cross-border e-commerce markets have shown unprecedented complexity and dynamics. According to the 2024 statistical report of the World Trade Organization, the global cross-border e-commerce transaction scale has exceeded 10 trillion U.S. dollars, with a compound annual growth rate of 19.3%. Transaction forms have evolved from the traditional B2C model to an ecosystem including B2B2C, social e-commerce, live-streaming shopping, and other diversified formats. The market data dimensions cover 32 dimensions such as transaction amount, commodity categories, space-time coordinates, and subject characteristics, forming a typical high-dimensional unstructured dynamic data flow. Its data generation mechanism has the dual characteristics of a stochastic process and a complex system. Traditional econometric methods face three core challenges in processing such data due to the lack of measure theory foundation: estimation bias caused by incomplete measures in high-dimensional spaces, failure of statistical inference due to the lack of σ-algebra closure in dynamic data flows, and compatibility contradictions between unstructured data and traditional measure frameworks.

As the cornerstone of modern analysis mathematics, measure theory provides a rigorous mathematical language for describing complex systems. Carathéodory's extension theorem can generate a complete measure space from the fundamental set structure through the hierarchical construction of semirings, premeasures, and outer measures, and its theoretical framework is naturally compatible with the multi-dimensional dynamic characteristics of cross-border e-commerce markets. However, existing studies have significant theoretical gaps in the following aspects: the measure system of the topological product space integrating time dimension and space dimension has not been constructed, the strict mathematical proof of the σ-finiteness of transaction data is lacking, and the market application verification for the completeness of null sets has not been formed. These theoretical gaps lead to the lack of a solid mathematical foundation for existing quantitative research on cross-border e-commerce, making it difficult to meet the requirements of market analysis accuracy in the digital economy era.

1.2. Research Significance

The theoretical contributions of this study are reflected in three aspects. First, it systematically introduces Carathéodory's extension theorem into the field of cross-border e-commerce for the first time, and constructs a measure system for the topological product space including the time index set T and the transaction state space X_t, filling the gap in basic theoretical research in this field. Second, through the axiomatic definition of σ-finite premeasure and the topological generation of outer measure, a measure extension mechanism suitable for high-dimensional dynamic data is established, providing a new theoretical paradigm for solving the measure completeness problem of unstructured data. Third, the completeness of the measure space is proved through strict mathematical derivation, and the basic property that "subsets of null sets must be measurable" is established, providing a theoretical guarantee for market microstructure analysis.

In terms of practical applications, this study has important value. At the methodological level, it provides standardized measurement tools for empirical research on cross-border e-commerce markets, which can be applied to scenarios such as measure value quantification of the gravity model and measure regularization of graph neural networks. At the policy analysis level, through the construction of the measure space, the marginal impact of policy and environmental factors such as tariffs and geographical distance on transaction measures can be accurately described, providing a scientific basis for cross-border e-commerce policy evaluation. At the technical innovation level, the integration of measure theory and machine learning provides new ideas for solving high-dimensional data modeling problems, and is expected to promote the paradigm upgrade of market analysis from traditional statistical methods to intelligent algorithms.

1.3. Research Framework and Technical Route

This study adopts a progressive research framework of “theoretical construction–mathematical proof–empirical verification”, and the specific technical route is as follows:

In the theoretical construction stage, first define the product space composed of the time index set T and the transaction state space X_t at each moment, where X_t is further decomposed into the Cartesian product of subspaces such as the transaction indicator space T_t, the amount space A_t, and the category space C_t. Secondly, construct a semiring structure S on the fundamental set, and realize the unified measure representation of continuous and discrete variables through the combination of a left-open right-closed interval semiring and a power set semiring. Finally, define a premeasure μ_0 that satisfies non-negativity, finite additivity, and σ-finiteness. Based on the outer measure μ^* and Carathéodory’s condition, the σ-algebra M is generated to complete the construction of the measure space.

The mathematical proof stage includes three aspects. First, the monotonicity, subadditivity, and normativity of the outer measure are strictly derived to lay the foundation for the definition of measurable sets. Second, the Carathéodory condition is used to prove that the family of measurable sets M constitutes a σ-algebra, confirming the uniqueness of measure extension. Third, the measurability of subsets of null sets is verified using outer measure theory to complete the mathematical proof of the completeness of the measure space.

The empirical verification stage is carried out based on cross-border e-commerce transaction data from 2018 to 2024. First, the Kolmogorov-Smirnov test is used to verify the distribution consistency between the theoretical measure and the empirical data. Second, a GMM panel regression model is constructed to quantitatively analyze the impact of factors such as GDP, geographical distance, and tariffs on transaction measures. Finally, a measure-aware graph neural network is designed to improve the model’s recognition ability for null sets through completeness regular terms, and verify the effectiveness of measure theory in algorithm optimization.

Through the organic combination of theory and empiricism, this study aims to establish a measure theory system for cross-border e-commerce markets that combines mathematical rigor and practical applicability, and provide a new analysis paradigm for international business research in the digital economy era.

2. Overview of Carathéodory’s Extension Theorem

2.1. Definition and Properties of a Semi-Ring

Let S be a non-empty collection of subsets on set X. If the following conditions are satisfied, then S is called a semi-ring on X:

Closure under finite intersections: If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$;

Finite decomposability of difference sets: If $A, B \in \mathcal{S}$ and $B \subseteq A$, then $A - B$ can be expressed as the union of a finite number of pairwise disjoint sets in S, that is, there exist $C_1, C_2, \cdots, C_n \in \mathcal{S}$ such that $A - B = \bigcup_{i=1}^{n} C_i$ and $C_i \cap C_j = \varnothing$ $(i \neq j)$.

Example: In the set of real numbers R, the collection $\mathcal{S} = \{(a, b] \mid a, b \in \mathbb{R}, a \leq b\}$ composed of all left-open and right-closed intervals is a semi-ring. For any $(a, b]$, $(c, d] \in \mathcal{S}$, the intersection $(a, b] \cap (c, d]$ can be expressed as $(\max(a, c), \min(b, d)]$, which still belongs to S; if $(c, d] \subseteq (a, b]$, then $(a, b] - (c, d] = (a, c] \cup (d, b]$ (when a < c < d ≤ b), which is the union of a finite number of pairwise disjoint semi-ring elements.

2.2. Definition of σ-Finite Pre-Measure

Let μ_0 be a set function defined on a semi-ring S, satisfying:

Non-negativity: For any $A \in \mathcal{S}$, $\mu_0(A) \geq 0$, and $\mu_0(\varnothing) = 0$;

Finite additivity: If $A_1, A_2, \cdots, A_n \in \mathcal{S}$ are pairwise disjoint and $\bigcup_{i=1}^{n} A_i \in \mathcal{S}$, then $\mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_0(A_i)$.

σ-Finiteness: There exists a countable collection of sets $\{X_i\}_{i=1}^{\infty} \subseteq \mathcal{S}$ such that $X = \bigcup_{i=1}^{\infty} X_i$, and for each i, $\mu_0(X_i) < +\infty$. Thenμ0 is called a σ-finite pre-measure on S.

2.3. Construction of Outer Measure

Based on the semi-ring S and the pre-measure μ_0, the outer measure $\mu^*: 2^X \to [0, +\infty]$ is defined as follows:

For any $E \subseteq X$, $\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(A_i) \,\middle|\, \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{S},\ E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$, where inf denotes the infimum, that is, the minimum value of the sum of the pre-measures of countable coverings of E by sets in S. The outer measure has the following basic properties:

Monotonicity: If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$;

Sub-additivity: For any countable collection of sets $\{E_i\}_{i=1}^{\infty} \subseteq X$, we have $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Normativity: For any $A \in \mathcal{S}$, $\mu^*(A) = \mu_0(A)$.

2.4. Measurable Sets and Measure Extension

Define the Carathéodory condition: A set $E \subseteq X$ is called a measurable set if for any $T \subseteq X$, there holds

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c)$$

where $E^c = X - E$ denotes the complement of E. The collection of all measurable sets is denoted as M. Then M is a σ-algebra, that is, it is closed under countable union, intersection, and complement operations.

On the measurable set system M, define the measure $\mu = \mu^*|_{\mathcal{M}}$, that is, μ is the restriction of the outer measure μ^* on M.

The Carathéodory extension theorem shows that: μ is the unique measure extension of μ_0 from the semi-ring S to the σ-algebra M, and the measure space $(X, \mathcal{M}, \mu)$ is complete, that is, the subset of a null-measurable set is measurable.

The theorem is precisely stated as: Let S be a semi-ring on set X, andμ0 be a σ-finite pre-measure on S. Then there exists a unique measure μ defined on the σ-algebra M containing S, such that $\mu|_{\mathcal{S}} = \mu_0$, and $(X, \mathcal{M}, \mu)$ is a complete measure space.

2.5. Theoretical Foundation of Cross-Border Market Measure Space

The theoretical underpinnings for constructing a measure space of cross-border markets lie in the structural isomorphism between the intrinsic properties of such markets and the axiomatic system of measure theory. Cross-border markets, as complex economic aggregates, exhibit a set-theoretic structure that admits formalization within the framework of measurable spaces, a fact rooted in their inherent capacity for systematic partitioning and aggregation. A primitive universe encompassing all transactional entities—agents, commodities, temporal modalities, regulatory constraints—serves as the foundational set, inherently supporting the derivation of a semi-ring structure from elementary market components. This semi-ring, characterized by closure under finite intersections and finite decomposability of set differences, provides the necessary algebraic scaffolding for measure construction, enabling the systematic extension of primitive “size” attributes to more complex market subsets.

The transition from semi-ring to σ-algebra, a cornerstone of measure space formation, is facilitated by the cross-border market’s intrinsic properties of countable concatenation and complementation. When categorized by intrinsic attributes, market elements exhibit closure under countable unions and complementation relative to the universal set, satisfying the defining axioms of σ-algebras. This closure ensures the resulting structure is amenable to rigorous manipulation of measurable sets, a prerequisite for meaningful measure-theoretic analysis. Critical to this framework is the existence of a σ-finite premeasure on the semi-ring—a non-negative, finitely additive set function encoding primitive “magnitudes” of market elements—which, by virtue of its σ-finiteness, admits unique extension to a measure on the generated σ-algebra via the Carathéodory construction, ensuring consistent “size” assignments across all measurable subsets of the market universe.

The completeness of the resulting measure space, a natural consequence of the extension process, resolves theoretical ambiguities posed by negligible market phenomena, as subsets of null-measure elements retain measurability. This completeness, coupled with the σ-finiteness of market observables, solidifies the cross-border market's status as a rigorously definable measure space, amenable to formal analysis through the full apparatus of measure theory.

3. Construction of Cross-Border E-Commerce Market Measure Space

3.1. Topological Algebra Construction of Fundamental Sets and Semirings

The basic set of cross-border e-commerce market is defined as a topological space, where T= is the set of time indicators and X_t represents the transaction state space at time t. Each X_t can be decomposed into a product space, where:

$$X = \prod_{t\in\mathbb{T}} X_t \mathbb{Z}^{+} X_t = \mathcal{T}_t \times \mathcal{A}_t \times \mathcal{C}_t \times \mathcal{R}_t \times \mathcal{P}_t$$

$T_t = \{0,1\}$ is the space of transaction occurrence indicators;

A_t is the transaction amount space, satisfying, M_t is the maximum transaction amount at time t; $\mathcal{A}_t = [0, M_t]$.

C_t is a finite set of commodity categories; $\mathcal{C}_t = \{c_1, c_2, \ldots, c_{n_t}\}$.

R_t is a finite set of trading regions; $\mathcal{R}_t = \{r_1, r_2, \ldots, r_{m_t}\}$.

P_t is the space of trading subjects, and S_t and B_t respectively represent the finite sets of sellers and buyers.

$$\mathcal{P}_t = \mathcal{S}_t \times \mathcal{B}_t$$

On the basic set X, construct a semi-ring structure S, defined as all cylinder sets of the form $A = \prod_{t\in\mathbb{T}} A_t$, where $A_t \in \mathcal{S}_t$, and except for finitely many t, $A_t = X_t$. Here, S_t is a semi-ring on X_t, and its specific construction is as follows:

For S_t, $\mathcal{S}_t^{\mathcal{T}} = \{\varnothing, \{0\}, \{1\}, \mathcal{T}_t\}$;

For A_t, $\mathcal{S}_t^{\mathcal{A}} = \{(a,b] \subseteq \mathcal{A}_t \,|\, a \le b\}$ is a semi-ring of left-open and right-closed intervals;

For C_t and R_t, $\mathcal{S}_t^{\mathcal{C}}$ and $\mathcal{S}_t^{\mathcal{R}}$ are respectively the semi-ring structures of power sets, that is, semi-rings generated by single-element sets;

For P_t, $\mathcal{S}_t^{\mathcal{P}} = \mathcal{S}_t^{\mathcal{S}} \times \mathcal{S}_t^{\mathcal{B}}$, where $\mathcal{S}_t^{\mathcal{S}}$ and $\mathcal{S}_t^{\mathcal{B}}$ are respectively the power set semi-rings of S_t and B_t.

Proof of the closure property of the semi-ring: Let $A = \prod_{t\in\mathbb{T}} A_t$, $B = \prod_{t\in\mathbb{T}} B_t \in \mathcal{S}$, then their intersection is $A \cap B = \prod_{t\in\mathbb{T}} (A_t \cap B_t)$. Since each S_t is closed under finite intersections, $A_t \cap B_t \in \mathcal{S}_t$, and except for finitely many t, $A_t \cap B_t = X_t \cap X_t = X_t$. Therefore, $A \cap B \in \mathcal{S}$, satisfying the closure property under finite intersections.

For the difference set operation, let $B \subseteq A$, then $A - B = \prod_{t\in\mathbb{T}} (A_t - B_t)$. Since each St satisfies the finite decomposability of difference sets, that is, there exist finitely many pairwise disjoint $C_{t1}, C_{t2}, \ldots, C_{tn_t} \in \mathcal{S}_t$ such that $A_t - B_t = \bigcup_{i=1}^{n_t} C_{ti}$. Therefore:

$$A - B = \prod_{t\in\mathbb{T}} \bigcup_{i=1}^{n_t} C_{ti} = \bigcup_{f\in F} \prod_{t\in\mathbb{T}} C_{tf(t)}$$

where F is a finite index set, and each $\prod_{t\in\mathbb{T}} C_{tf(t)}$ is an element in S and pairwise disjoint, thus proving the finite decomposability of the difference set.

3.2. Axiomatic Definition of Premeasure and Proof of σ-Finiteness

On the semi-ring S, a pre-measure $\mu_0 : \mathcal{S} \to [0, +\infty]$ is defined. For any $A = \prod_{t\in\mathbb{T}} A_t \in \mathcal{S}$, let $T_A = \{t \in \mathbb{T} \,|\, A_t \neq X_t\}$ be the non-trivial coordinate set, and its cardinality $|T_A| < +\infty$. Define:

$$\mu_0(A) = \sum_{t\in T_A} \int_{A_t} w_t(x_t)\, d\lambda_t(x_t)$$

where:

w_t: $X_t \to \mathbb{R}_+$ is the weight function of transaction amount. Specifically, $w_t(x_t) = \begin{cases} a & \text{if } x_t \in \mathcal{T}_t \times \{a\} \times \cdots \\ 0 & \text{else} \end{cases}$, that is, it extracts the transaction amount component; λ_t is the Lebesgue measure or counting measure on R_t, determined by the topological properties of R_t: for A_t, the Lebesgue measure is used, and for discrete spaces T_t, C_t, R_t, P_t, the counting measure is used.

Axiomatic verification of pre-measure:

(1) Non-negativity: Since $w_t(x_t) \geq 0$ and the measure λ_t is non-negative, $\mu_0(A) \geq 0$, and $\mu_0(\varnothing) = 0$ obviously holds.

(2) Finite additivity: Let $\{A_i\}_{i=1}^n \subseteq \mathcal{S}$ be pairwise disjoint, and $A = \bigcup_{i=1}^n A_i \in \mathcal{S}$. For each $t \in \mathbb{T}$, $\{A_{it}\}_{i=1}^n$ are pairwise disjoint sets in X_t, and $\bigcup_{i=1}^n A_{it} = A_t$ (when $t \in T_A$) or X_t (when ($t \notin T_A$). According to the finite additivity of the measure, we have [1]:

$$\mu_0(A) = \sum_{t \in T_A} \int_{A_t} w_t(x_t) d\lambda_t(x_t) = \sum_{t \in T_A} \sum_{i=1}^n \int_{A_{it}} w_t(x_t) d\lambda_t(x_t) = \sum_{i=1}^n \mu_0(A_i)$$

(3) σ-Finiteness: Construct a countable covering $\{X^{(k)}\}_{k=1}^\infty$, where $X^{(k)} = \prod_{t=1}^k X_t \times \prod_{t=k+1}^\infty X_t$, that is, take the full space at the first k time points and also take the full space at subsequent time points. Obviously, $X = \bigcup_{k=1}^\infty X^{(k)}$, and for each k, $X^{(k)} \in \mathcal{S}$. Calculate the pre-measure:

$$\mu_0(X^{(k)}) = \sum_{t=1}^k \int_{X_t} w_t(x_t) d\lambda_t(x_t)$$

Since the transaction amount space A_t at each time t is bounded, that is

$$\int_{X_t} w_t(x_t) d\lambda_t(x_t) = \int_{\mathcal{A}_t} a d\lambda_t(a) \leq M_t \cdot \lambda_t(\mathcal{A}_t) < +\infty$$

Therefore $\mu_0(X^{(k)}) < +\infty$, it is proved that μ_0 is a σ-finite predictor.

3.3. Topological Generation of Outer Measure and Measurable Sets

Based on the predicted quantity μ0, the outer measure is defined as: μ^*: $2^X \to [0, +\infty]$

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^\infty \mu_0(A_i) \,\middle|\, \{A_i\}_{i=1}^\infty \subseteq \mathcal{S},\ E \subseteq \bigcup_{i=1}^\infty A_i \right\}$$

Among them, the lower bound traverses all countable covers composed of semi-ring elements [2]. The outer measure has clear economic significance in the cross-border e-commerce market scene. Its essence is to use the linear combination of basic transaction units to approximate the "total transaction volume" lower bound of any market state set.

According to Carathéodory's condition, a measurable set is defined as satisfying for any, with: $\mathcal{M} \subseteq 2^X T \subseteq X$.

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c)$$

It can be proved that M is a σ-algebra containing the semi-ring S, and the measure $\mu = \mu^*|_{\mathcal{M}}$ defined on M is the unique measure extension of μ_0. Thus, a complete measure space $(X, \mathcal{M}, \mu)$ is constructed. This construction makes any observable event in the cross-border e-commerce market, such as the transaction set of a certain type of commodity in a certain region within a specific time period, correspond to a measurable set in the measure space, and its measure value is the total transaction volume of this event, providing a strict mathematical basis for subsequent quantitative analysis [3].

4. Proof of Completeness of Cross-Border E-Commerce Market Measure

4.1. Prerequisite Knowledge and Symbol Explanation

Before proving completeness [4], several key properties and symbols are clarified. For any sets $A, B \subseteq X$, if $A \subseteq B$, according to the definition of the outer measure, we know that $\mu^*(A) \leq \mu^*(B)$.

For a measurable set $E \in \mathcal{A}$, it satisfies: $\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c), \forall T \subseteq X$; meanwhile, the measure μ is the restriction of the outer measure μ* on the measurable set A, that is, for $E \in \mathcal{A}$, $\mu(E) = \mu^*(E)$.

4.2. Proof Process of Completeness

The measure space $(X, \mathcal{A}, \mu)$ is defined to be complete if $N \in \mathcal{A}$, and $\mu(N) = 0$, and $M \subseteq N$, then $M \in \mathcal{A}$, and $\mu(M) = 0$. Moreover, for any $M \subseteq N$ and any $T \subseteq X$, the following steps are used to prove that:

(1) Derivation of outer measure relation:

Because $M \subseteq N$, according to the monotonicity of the outer measure (if $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$), for any $T \subseteq X$, there is $T \cap M \subseteq T \cap N$, so $\mu^*(T \cap M) \leq \mu^*(T \cap N)$.

And because$N \in \mathcal{A}$, according to the definition of measurable sets, for any $T \subseteq X$, there is $\mu^*(T) = \mu^*(T \cap N) + \mu^*(T \cap N^c)$.

It is known that$\mu(N) = \mu^*(N) = 0$ for any$T \subseteq X$, let $T = N$, then, $\mu^*(N) = \mu^*(N \cap N) + \mu^*(N \cap N^c) = \mu^*(N) + \mu^*(\emptyset) = 0$, so, $\mu^*(\emptyset) = 0$.

Since $T \cap N \subseteq N$, again according to the monotonicity of the outer measure $\mu^*(T \cap N) = 0$, it follows that $\mu^*(T \cap M) = 0$.

(2) Verify the Carassio-Dorri condition:

According to the subadditivity of the outer measure, for any $T \subseteq X$, there is $\mu^*(T) \leq \mu^*(T \cap M) + \mu^*(T \cap M^c)$.

Because it has been proved $\mu^*(T \cap M) = 0$, so $\mu^*(T) \leq \mu^*(T \cap M^c)$.

And because $T \cap M^c \subseteq T$, according to the monotonicity of the outer measure, $\mu^*(T \cap M^c) \leq \mu^*(T)$.

In conclusion, $\mu^*(T) = \mu^*(T \cap M) + \mu^*(T \cap M^c)$, the Kardasseyo condition is satisfied, so $M \in \mathcal{A}$.

4.3. Proof

According to the non-negativity of the outer measure, there is $0 \leq \mu^*(M) \leq \mu^*(N) = 0$, so $\mu^*(M) = 0$ [5,6].

Because $M \in \mathcal{A}$, and the measure μ is the restriction of the outer measure μ* on the measurable set $\mathcal{A}$, that is $\mu(M) = \mu^*(M)$, so $\mu(M) = 0$. In conclusion, for any subset M of a null set N in the measure space $(X, \mathcal{A}, \mu)$, we have $M \in \mathcal{A}$ and $\mu(M) = 0$, it is proved that the measure space is complete.

5. Empirical Analysis of Cross-Border E-Commerce Market Measure

5.1. Source and Preprocessing of Empirical Data

5.1.1. Data Collection Framework

The data in this paper is sourced from three major channels, constructing a three-dimensional dataset incorporating micro-transactions, macroeconomics, and market environment. The time span is from January 2018 to December 2024, covering major global cross-border e-commerce markets:

(1) Micro-transaction data: Original transaction records from 10 major platforms (Amazon, eBay, AliExpress, etc.) are obtained. After cleaning, 230 million valid records are retained, covering 32 dimensions such as transaction amount, commodity categories, countries/regions of both transaction parties, transaction timestamps, and logistics methods.

(2) Macroeconomic data: Quarterly GDP, bilateral exchange rates, import tariff rates, and trade policy indices of 150 countries released by institutions like the World Bank and IMF are collected.

(3) Market environment data: Including platform policy change logs (e.g., commission adjustments, logistics subsidies), international logistics cost indices, consumer confidence indices, and construction based on Google Trends search popularity.

5.1.2. Data Preprocessing Process

A multi-level data processing scheme based on measurement theory is adopted to ensure that the data meets the measurability requirements of the measurement space:

(1) Missing Value Processing: For key variables such as transaction amounts, utilize the monotonicity of the outer measure proved in Section 4. Through $\mu^*(A) \leq \mu^*(B)$ (when $A \subseteq B$), construct an interval of values.

Combine with multiple imputation methods to generate 10 sets of imputed values. Finally, use the measure estimate $\hat{\mu}(A)$ as the imputation result.

(2) Outlier Detection: According to the definition of measure completeness, transactions with a measure $\mu(N) < 10^{-6}$ are defined as outliers. First, identify amount outliers using the 3σ principle, and then use the Isolation Forest algorithm to detect outliers in dimensions such as category and region. Finally, 0.23% of the outlier records are removed.

(3) Data standardization: the continuous variables are standardized by quantile to make the data obey [0, 1] uniform distribution; for discrete variables (such as category and region), the unique hot coding based on measurement frequency is adopted to ensure that the coded variables satisfy the countable operation closure of σ-algebra;

(4) Spatiotemporal aggregation: The transaction data are aggregated according to three dimensions: quarter (T = 28 periods), country/region (R = 100) and commodity category (C = 50) to construct the panel data set $28 \times 100 \times 50$. The aggregation is carried out in accordance with the countable additivity of measurement $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ (when E_i are not intersecting).

5.1.3. Measurement Consistency Test

Outer Measure Matching Test

Select the typical measurable set A = {transactions with transaction amounts [3000, 7000] yuan and categories {c2, c5}. This set is constructed by considering both the continuous range of transaction amounts and the discrete selection of product categories, reflecting the typical continuous-discrete hybrid structure in cross-border e-commerce transaction data.

The core logic of the test is to measure the consistency of the distribution between the empirical measure $\hat{\mu}(A)$ and the theoretical measure $\mu(A)$ through the Kolmogorov-Smirnov (K-S) test. As a non-parametric test method, the K-S test does not require presupposing the distribution form of the data. Its theoretical basis originates from the difference measurement between the empirical distribution function and the theoretical distribution function. For real-valued data with a sample size of n, the empirical distribution function $F_n(x)$ is defined as $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x)$ (where$I(\cdot)$ is an indicator function); the theoretical distribution function $F(x)$ is generated by the theoretical measure of the measurable set A in the measure space. The expression of the K-S test statistic is:

$$D = \sup_x \left| F_n(x) - F(x) \right|$$

This statistic characterizes the maximum deviation between empirical distribution and theoretical distribution. By comparing with the critical value of K-S distribution, the null hypothesis that "empirical distribution is consistent with theoretical distribution" can be tested.

The K-S test was conducted on the empirical $\hat{\mu}(A)$ and theoretical measures $\mu(A)$ of the measurable set A, yielding a K-S statistic of $D = 0.032$, with a corresponding significance level $p = 0.12$. This result does not reject the null hypothesis that the empirical and theoretical measures are consistent in distribution. This indicates that, on this typical measurable set, the theoretical measure space constructed based on the Caratheodory extension theorem can effectively capture the distribution characteristics of cross-border e-commerce empirical transaction data. The outer measure demonstrates good matching in real-world transaction scenarios, validating the theoretical frameworks adaptability to continuous-discrete mixed structure data.

Completeness Verification

In a measure space, completeness requires that any subset of the null set is measurable and has a measure of 0. To verify the completeness of the theoretical measure space in empirical data, construct the null set N = {transactions with a transaction amount of 10,000 yuan} and its subset M = {transactions with a transaction amount of 10,000 yuan and a seller s1}.

From the perspective of the actual scenario of cross-border e-commerce transactions, the event that "the transaction amount is exactly a certain specific value" has a low probability of occurrence and can be regarded as a null-measure set in the theoretical measure space (that is, $\mu(N)=0$; and the subset M further defines the seller characteristics of the transaction, being a more refined subset of the null-measure set N. Its theoretical measure should also satisfy $\mu(M)=0$ and must conform to the monotonicity of the outer measure ($\mu(M)\le\mu(N)$). Through the empirical processing of cross-border e-commerce transaction data, the null-measure set N is estimated.

By empirically processing cross-border e-commerce transaction data, the core of estimating the measures of the zero-measure set N and the subset M involves approximating the measure through transaction frequency. First, the transaction data set from the cross-border e-commerce platform is collected, which includes key fields such as transaction amounts and seller information. Then, the number of transactions that meet the conditions of N and M is counted, with nN representing the number of transactions meeting the condition of N and nM representing the number of transactions meeting the condition of M. Based on this, the empirical measure is defined as the proportion of transactions that meet the conditions, which serves to approximate the measure of the zero-measure set. The results are as follows:

Empirical measure of null set N: $\hat{\mu}(N)=0.0005$.

Empirical measure of subset M: $\hat{\mu}(M)=0.000012$.

In terms of numerical characteristics, both are close to the theoretical measure value of 0, reflecting the characteristics of "low probability and low measure" of null-measure sets and their subsets in actual transactions. Meanwhile, the empirical measure satisfies $\hat{\mu}(M)\le\hat{\mu}(N)$, which is completely consistent with the monotonicity of the outer measure (if $M\subseteq N$), then $\mu(M)\le\mu(N)$.

This result shows that in real transaction data, the measurement performance of zero test sets and their subsets is highly consistent with the definition of completeness of theoretical measurement space.

5.2. GMM Panel Regression Model

5.2.1. Model Building

Based on the measurement space structure in Section 3 and the completeness theory in Section 4, we construct a GMM [7,8] two-way fixed effect panel regression model. The expression of the model is as follows:

$$\begin{aligned}&\ln\mu(A_{ijtc})=\beta_0+\beta_1\ln GDP_i(t)+\beta_2\ln GDP_j(t)+\beta_3\ln Dist_{ij}+\\&\beta_4 Tariff_{ij}(t)+\beta_5 ERP_{ij}(t)+\beta_6 CAT_c+\delta_t+\eta_{ij}+\epsilon_{ijtc}\end{aligned}$$

Among them: $\mu(A_{ijtc})$ is the measure value of "transactions of category c from country i to country j in period t" in the measure space, estimated by

$$\hat{\mu}(A)=\frac{1}{n}\sum_{k=1}^{n}\mathbb{I}(X_k\in A)\cdot w(X_k)$$

$GDP_i(t)$, $GDP_j(t)$ are the quarterly GDP of country i and country j respectively (in constant US dollars); $Dist_{ij}$ is the geographical distance (in kilometers) between the capitals of country i and country j; $Tariff_{ij}(t)$ is the average import tariff rate (%) of country i to country j in period t; $ERP_{ij}(t)$ is the policy index of the e-commerce platform (0–100), and a larger value indicates a greater support from the platform for this trading pair; CAT_c is the category effect, reflecting the basic transaction level of different categories; δ_t is the time-fixed effect, η_{ij} is the region-fixed effect, and ϵ_{ijtc} is the random error term, satisfying $\mu(\{\epsilon_{ijtc}>\sigma\})$.

5.2.2. Model Estimation

The system GMM estimation method is adopted to deal with the potential endogeneity problem, and the instrumental variables are selected from the GDP and tariff variables lagged by two periods. The estimation results are shown in Table 1.

Table 1. GMM estimation results table.

Variable	Coefficient	Standard Error	*t* Price	*p* Price
$\ln GDP_i(t)$	0.721 **	0.035	20.60	0.000
$\ln GDP_j(t)$	0.683 **	0.032	21.34	0.000
$\ln Dist_{ij}$	−0.547 **	0.028	−19.54	0.000
$Tariff_{ij}(t)$	−1.235 **	0.057	−21.67	0.000
$ERP_{ij}(t)$	0.812 **	0.041	19.80	0.000
CAT_c (Reference category c1)	0.215 *	0.098	2.19	0.029
time effect	Its under control	-	-	-
Regional effects	Its under control	-	-	-
sample capacity	140,000	-	-	-

The marker * denotes that the estimated regression coefficient achieves statistical significance at the 10% significance level. The ** marker signals significance at the 5% significance level.

5.2.3. Result Analysis

Dynamic Correlation Mechanism between Economic Scale and Cross-Border E-Commerce Transactions

As shown in the estimation results of Table 1, the coefficient of the natural logarithm of the importing countrys GDP is 0.683, which passes the 1% significance test, and the coefficient of the natural logarithm of the exporting countrys GDP is 0.721, also significant at the 1% level. This indicates a significant positive elasticity relationship between cross-border e-commerce transaction measures and the economic aggregates of both countries. Specifically, for every 1% increase in the importing countrys GDP, the transaction measure increases by 0.683%, while for every 1% increase in the exporting countrys GDP, the transaction measure increases by 0.721%. The marginal impact of the exporting countrys GDP on the transaction measure is 5.56% higher than that of the importing country. This difference can be explained from the dual dimensions of supply and demand in international trade theory: under the traditional trade framework, the industrial supply capacity of the exporting country is often the core factor determining the scale of trade. However, the digital empowerment of cross-border e-commerce platforms further strengthens this mechanism—Export companies can leverage the global market reach of e-commerce platforms to expand their supply at lower marginal costs, with the economies of scale being more pronounced in the digital trade environment. Additionally, the information aggregation function of e-commerce platforms effectively reduces information asymmetry in international trade, enabling high-quality supply to bypass traditional trade barriers and directly connect with the global consumer market, thus forming a stronger pull effect of the exporting countrys economic scale on cross-border transactions answer.

Analysis of the Alienation Effect of Geographical Distance on Cross-Border E-Commerce Transactions

The estimated coefficient for the natural logarithm of geographical distance is −0.547, which is statistically significant at the 1% level. This value is smaller in absolute terms than the typical estimate of about −0.8 in traditional gravity models of trade, indicating that cross-border e-commerce is less sensitive to geographical distance compared to traditional trade. This phenomenon reflects three key transformations in digital trade: Firstly, the digital upgrade of the modern logistics system has completely transformed the spatial constraints of traditional trade. The Amazon FBA model, which uses pre-warehousing, has transformed international logistics into regional domestic logistics, significantly reducing delivery times and lowering transportation uncertainties, thereby decreasing the actual transaction cost impact of geographical distance by approximately 31.6%. Secondly, the development of Internet technology has made the transmission of transaction information transcend time and space limitations, reducing communication costs between buyers and sellers to nearly zero. As a result, the importance of geographical proximity as the primary factor in choosing trading partners has been significantly diminished, and the reduction in information costs has directly weakened the inhibitory effect of geographical distance on transactions. Thirdly, the rise of new trading models such as B2B2C and social e-commerce has disrupted the hierarchical distribution system of traditional trade, enabling direct connections between producers and consumers. The flattened transaction chain has reduced the geographical dependence of intermediate links, further diminishing the marginal impact of geographical distance. From the perspective of spatial economics, this result shows that cross-border e-commerce is reshaping the geographical pattern of global trade. The “distance decay law” in traditional trade shows a phased weakening trend in the digital trade environment, which provides a new window of opportunity for developing countries with remote geographical locations to participate in the global value chain.

The Asymmetric Impact Mechanism of Tariff Policy on Cross-Border E-Commerce Transactions

The estimated coefficient of the tariff rate variable is −1.235, which passes the test at the 1% significance level. The absolute value of this coefficient is not only significantly higher than the elasticity of geographical distance but also 2.26 times greater, highlighting the critical role of tariff policies in cross-border e-commerce transactions. A deeper analysis reveals three key mechanisms: First, the amplification effect of price transmission. Higher tariffs directly increase the tax-inclusive cost of imported goods. In the cross-border e-commerce market, where consumers are highly price-sensitive and competition is intense, the price increases due to tariffs can significantly dampen consumer demand. This effect is further amplified by the transparent pricing mechanisms of e-commerce platforms. Second, the heterogeneous response of trading entities. Most cross-border e-commerce participants are small and medium-sized enterprises (SMEs) and individual consumers, who have a much lower tolerance for tariff costs compared to large enterprises in traditional trade. A slight increase in tariff rates can lead to a significant number of small and medium sellers exiting the market, creating a pronounced cliff effect that non-linearly suppresses transaction volumes. Finally, the policy substitution effect. The high elasticity of tariffs reflects the high sensitivity of cross-border e-commerce to policy environments. When tariffs are tightened, market players may switch to other more lenient trade channels or markets, exacerbating this substitution behaviorIt has a negative impact on the measurement of transactions. In general, tariff policy plays a “super marginal” regulatory role in cross-border e-commerce trade, and its influence on transaction size is far greater than traditional factors such as geographical distance, which has become the main policy obstacle to the development of cross-border e-commerce.

Analysis of the Interaction between Category Heterogeneity and Platform Policy

As shown in the fixed effects of categories in Table 1, using the basic category c1 as a reference, the transaction measurement for the electronic products category c2 is 21.5% higher and passes the 5% significance test. This result confirms the trading advantage of high-value-added goods in cross-border e-commerce. The reason for this is that high-value-added goods typically have higher demand elasticity and brand premium potential, which can better cover the logistics and operational costs of cross-border e-commerce. Additionally, the display and evaluation mechanisms on e-commerce platforms are more conducive to realizing the value of high-value-added goods. Moreover, the estimated coefficient of the platform policy index is 0.812, significant at the 1% level, indicating that for every one-unit increase in platform support, the transaction measurement increases by 0.812%. This reflects the significant role of e-commerce platforms institutional arrangements in promoting cross-border transactions. Platforms can effectively reduce transaction costs and enhance market efficiency through policy tools such as commission adjustments and logistics subsidies. This internal policy support from platforms contrasts with national-level tariff policies, highlighting the adaptive adjustment capabilities of market entities in response to trade policy environments.

5.2.4. Robustness Test

In order to ensure the reliability and universality of the empirical results, this study conducts systematic robustness tests from four dimensions: model setting, sample range, variable measurement and endogeneity treatment, and adopts rigorous statistical methods to verify the benchmark regression results in multiple dimensions.

Model Setting Level

By comparing the system GMM and difference GMM estimation methods, it is found that the difference GMM addresses endogeneity issues by eliminating individual fixed effects through first-order differencing. The system GMM combines level and difference equations and introduces lagged variables as instrumental variables, which not only enhances estimation efficiency but also effectively controls for endogeneity. The estimation results show that the signs and significance levels of the core explanatory variables have not changed substantially. Specifically, the log GDP coefficient of the importing country is 0.679 ($p < 0.01$) and 0.683 ($p < 0.01$) under the difference GMM and system GMM, respectively. The log GDP coefficient of the exporting country is 0.718 ($p < 0.01$) and 0.721 ($p < 0.01$), and the log geographical distance coefficient is −0.543 ($p < 0.01$) and −0.547 ($p < 0.01$). Additionally, the tariff rate coefficient is −1.231 ($p < 0.01$) and −1.235 ($p < 0.01$). These findings indicate that the model estimation results are robust across different GMM estimation methods.

Sample Scope

Considering the heterogeneous characteristics of the cross-border e-commerce market at different stages of development, the full sample from 2018 to 2024 is divided into two sub-periods: the growth period from 2018 to 2021 and the maturity period from 2022 to 2024, for separate estimation. During the growth period, cross-border e-commerce experienced rapid expansion in scale, while the maturity period saw structural optimization and business model upgrades. The estimation results show that during the growth period, the log coefficient of the importing countrys GDP is 0.692 ($p < 0.01$), the exporting countrys GDP is 0.735 ($p < 0.01$), geographical distance is −0.538 ($p < 0.01$), and tariff is −1.246 ($p < 0.01$). In the maturity period, these coefficients are 0.678 ($p < 0.01$), 0.712 ($p < 0.01$), −0.552 ($p < 0.01$), and −1.228 ($p < 0.01$), respectively. Although there are slight differences in the coefficients across stages, the direction and significance of the core variables remain stable, indicating that the research conclusions are not significantly affected by the choice of the sample period.

Variable Robustness

This study adjusted the measurement methods for key explanatory variables: converting GDP data from constant price dollars to PPP-adjusted values, and recalculating the logarithms of GDP for each country. It also used the trade policy uncertainty index as a substitute for traditional tariff rate variables to more comprehensively capture the dynamic impact of the trade policy environment. The re-estimation results showed that the coefficient of the logarithm of the GDP of the importing country, after PPP adjustment, was 0.685 ($p < 0.01$), the coefficient of the logarithm of the exporting countrys GDP was 0.723 ($p < 0.01$), the coefficient of the logarithm of geographical distance was −0.549 ($p < 0.01$), and the coefficient of the trade policy uncertainty index was −1.352 ($p < 0.01$). These results are highly consistent with the baseline regression, confirming the robustness of the conclusions regarding the measurement methods of the variables.

Endogeneity of the Model

The core explanatory variable, lagged by three periods, was used as an instrumental variable for the system GMM estimation. The *p*-value of the over-identification test is 0.234, indicating that the selection of the instrumental variables meets the exogeneity condition. The serial correlation test shows that the *p*-value for AR (1) is 0.000 (rejecting the null hypothesis of no autocorrelation), and the *p*-value for AR(2) is 0.123 (accepting the null hypothesis of no second-order autocorrelation), which aligns with the requirements for system GMM estimation. The coefficients and significance levels of the core explanatory variables have not changed substantially after adjustment, further confirming that the model does not suffer from severe endogeneity issues, ensuring the reliability of the estimation results.

5.2.5. Summary

This section, based on the constructed cross-border e-commerce market measurement space, uses a GMM panel regression model to quantitatively analyze the impact of factors such as GDP, geographical distance, and tariffs on transaction measurements. The model estimation results show that the GDP of both importing and exporting countries has a significant positive effect on transaction measurements, while geographical distance and tariffs have a restraining effect, with the impact of tariffs being much greater than that of geographical distance. After conducting robustness tests using various methods, including different model settings, sample divisions, variable measurements, and endogeneity treatments, the results all confirm the reliability of the benchmark model.

5.3. Empirical Analysis of Machine Learning Models

5.3.1. Architectural Design and Theoretical Logic of Measure-Aware Graph Neural Network

To achieve organic integration of measure theory and deep learning, this study constructs a Graph Neural Network model (MeasureGNN) with measure constraint capabilities. The model takes countries/regions as basic node units and bilateral transaction measure values as edge weights to build a directed weighted graph structure G = (V,E), where edge weights strictly follow the non-negativity and monotonicity characteristics of measures, ensuring the graph structure accurately maps transaction correlations in cross-border e-commerce markets. Node features are extracted from the micro-panel data of cross-border e-commerce transactions from 2018 to 2024. After preprocessing flows such as missing value imputation and outlier removal, a three-dimensional feature matrix including quarters, countries/regions, and product categories is generated, providing the model with input information combining temporal dynamics and spatial correlation.

In the design of core network layers, the measure propagation layer adopts multi-layer graph convolution operations, whose mathematical expression is:

$$h_v^{(l+1)} = \sigma\left(\sum_{u \in N(v)} \frac{1}{\sqrt{|N(v)||N(u)|}} h_u^{(l)} W^{(l)}\right)$$

where $h_v^{(l)}$ represents the feature vector of node v at the l-th layer, N(v) is the set of neighbor nodes of node v, $W^{(l)}$is the learnable weight matrix, and σ is the activation function. Notably, the model introduces a standardization weight mechanism based on measure theory. Through normalization processing of edge weights, it ensures that information propagation strictly satisfies the countable additivity principle of measures, avoiding measure estimation bias caused by weight imbalance.

Additionally, the model innovatively incorporates a completeness regular term:

$$\mathcal{L}_{reg} = \lambda \sum_{M \subseteq N, \mu(N)=0} \left(\hat{\mu}(M)\right)^2$$

This regular term directly reflects the completeness property of the measure space proved in Section 4 by forcing the measure estimation values of subsets of null sets to approach 0, effectively suppressing the overfitting phenomenon of the model for low-probability transaction events and improving the ability to identify null sets and their subsets.

5.3.2. Model Training Strategies and Computational Implementation Details

To ensure the stability and generalization capability of model training, this study adopts a 10-fold cross-validation evaluation scheme, dividing the dataset into training, testing, and validation sets at a ratio of 7:2:1. In the design of the optimization objective, a combined loss function is used:

$$\mathcal{L} = \alpha\mathcal{L}_{mse} + \beta\mathcal{L}_{l1} + \gamma\mathcal{L}_{reg}$$

where $\mathcal{L}_{mse}$ is the mean squared error loss for measuring the overall deviation between predicted and real measures; $\mathcal{L}_{l1}$ is the mean absolute error loss to enhance robustness against extreme values; $\mathcal{L}_{reg}$ is the completeness regular term mentioned above. The weight coefficients are set α = 0.5, β = 0.3, and γ = 0.2, achieving the optimal balance between model accuracy and theoretical consistency by balancing the contributions of different loss terms.

The hyperparameter optimization links adopt the Bayesian optimization algorithm. After multiple iterations, the optimal network architecture is determined: a three-layer structure with hidden layer dimensions of 256-128-64, a learning rate of 0.001, an L2 regularization coefficient of 0.001, and a dropout rate of 0.3. The model training relies on 4 NVIDIA A100 GPUs for parallel computing, with a single training epoch taking approximately 15 min. Batch

Normalization technology is used to alleviate the gradient vanishing problem, combined with the Early Stopping strategy to avoid overfitting in the training process. The following is a core Python code snippet of the model implementation, clearly demonstrating the specific realization of the measure constraint mechanism in the network layer:

```
import torch
import torch.nn as nn
import torch.nn.functional as F
# Define the Measure-Aware Graph Neural Network model
class MeasureGNN(nn.Module):
    def __init__(self, node_features, hidden_channels, out_channels, dropout = 0.3):
        super(MeasureGNN, self).__init__()
        # Measure propagation layer (GCN convolution) for multi-order neighborhood information aggregation through three convolution layers
        self.conv1 = gnn.GCNConv(node_features, hidden_channels)
        self.conv2 = gnn.GCNConv(hidden_channels, hidden_channels)
        self.conv3 = gnn.GCNConv(hidden_channels, out_channels)
        # Measure standardization weight layer to learn the measure mapping relationship of edge weights through an MLP network
        self.edge_mlp = nn.Sequential(
            nn.Linear(1, 64),
            nn.ReLU(),
            nn.Linear(64, 1)
        )
        # Activation function and regularization modules to improve model generalization
        self.relu = nn.ReLU()
        self.dropout = nn.Dropout(dropout)
        self.batch_norm = nn.BatchNorm1d(hidden_channels)
    def forward(self, x, edge_index, edge_attr):
        # Edge weight standardization: constrained by the countable additivity of measure theory
        edge_weights = torch.sigmoid(self.edge_mlp(edge_attr))
        # Normalization processing to ensure the sum of edge weights is 1, conforming to the additivity principle of measures
        edge_weights = edge_weights/(torch.sum(edge_weights, dim = 0, keepdim = True) + 1e-8)

        # Multi-layer measure propagation process for layer-wise aggregation of neighborhood transaction measure information
        x = self.conv1(x, edge_index, edge_weight = edge_weights)
        x = self.batch_norm(x)
        x = self.relu(x)
        x = self.dropout(x)
        x = self.conv2(x, edge_index, edge_weight = edge_weights)
        x = self.batch_norm(x)
        x = self.relu(x)
        x = self.dropout(x)
        x = self.conv3(x, edge_index, edge_weight = edge_weights)
        return x.squeeze()
```

5.3.3. Model Performance Evaluation and Comparative Analysis

In terms of measure estimation accuracy, the model achieves a Mean Squared Error (MSE) of $9.8 \times 10^4$and a Mean Absolute Error (MAE) of 765.32 yuan on the validation set, demonstrating precise characterization of transaction measures. Comparing the performance of traditional GNN models, Random Forest models, and Gradient Boosting Tree models (as shown in Table 2), the measure-aware graph neural network shows significant advantages on training, validation, and test sets: the training set MSE is reduced by 35.6% compared with traditional GNN, the validation set MAE is reduced by 24.4%, and the test set MSE is improved by approximately 35.8% compared with traditional GNN models. This result fully verifies the effectiveness of integrating measure theory with graph neural networks. In particular, the introduction of the completeness regular term significantly improves the model's ability to handle low-probability transaction events such as null sets, avoiding estimation biases in traditional models due to the lack of measure constraints.

Table 2. Comparative Results of Model Performance Evaluation**.**

Types of Models	Data Set	MSE	MAE
Measurement of perceptual graph neural network	training set	8.7×10^4	682.5
	validation set	9.8×10^4	765.3
	test set	1.02×10^6	792.1
GNN model	training set	1.35×10^5	925.7
	validation set	1.48×10^5	1012.4
	test set	1.55×10^6	1056.2
Random forest model	training set	1.12×10^5	853.6
	validation set	1.21×10^5	901.3
	test set	1.28×10^6	937.5
Gradient boosting tree model	training set	1.05×10^5	812.8
	validation set	1.13×10^5	864.2
	test set	1.19×10^6	898.6

5.3.4. Model Interpretability Analysis Based on SHAP Values

To deeply understand the contribution of each feature in the model's decision logic, this study uses the SHAPvalue method for interpretability analysis. The results show that among the key features affecting transaction measure estimation, the mean SHAP value of importing country GDP is 0.283, and that of exporting country GDP is 0.275, indicating that economic scale is the core factor driving transaction measures. The SHAP value of tariff rate is −0.312, with the largest absolute value, confirming the strong inhibitory effect of tariff policies on cross-border transactions. The SHAP value of geographical distance is −0.214, reflecting the weakening effect of digital trade on spatial constraints. The SHAP value of the platform policy index is 0.187, indicating that the support policies of e-commerce platforms can effectively promote the improvement of transaction measures. This conclusion is highly consistent with the results of the GMM panel regression, which not only verifies the reliability of the model but also further confirms the effectiveness of measure theory in describing the dynamic relationships in cross-border e-commerce markets from the perspective of machine learning.

6. Conclusions

6.1. Core Theoretical Contributions

This study successfully applied the Karchiorod expansion theorem to construct a complete measurement space (X, M, μ) of cross-border e-commerce market, and its core innovation points are as follows:

(1) Spatial structure: The market base set is defined as a topological product space $X = \prod_{t \in T} X_t$, and a semi-

ring S is constructed by column sets. In which the transaction amount space adopts a left-open and right-closed interval semi-ring, and the discrete dimension adopts a power set semi-ring, so as to realize the structured representation of multi-dimensional market data;

(2) Measure expansion: Define a σ-finite predictive measure μ_0 on the semicircle S. By using Lebesgue and counting measures to characterize the trading weights across different dimensions, and through the outer measure μ^* and the Kallar-Scholze condition, construct the σ-algebra M. This proves that the measure μ is the unique complete extension of μ_0;

(3) Completeness proof: Through the verification of the monotonicity, subadditivity and Karchiorodri condition of zero-measure subsets of the outer measure, it is confirmed that "the subset of zero-measure must be measurable" in the measure space, which provides a theoretical guarantee for market quantitative analysis.

6.2. Empirical Application Value

Based on the constructed measurement space, this study carried out empirical analysis and model construction, and drew the following key conclusions:

(1) Data compatibility: Through Kolmogorov-Smirnov test (D = 0.032, p = 0.12) and zero set verification; ($\mu(N) \approx 0$, $\mu(M) \approx 0$), it is confirmed that the theoretical measurement is highly consistent with the cross-border e-commerce data from 2018 to 2024;

(2) GMM Model Analysis: For the importing country, a 1% increase in GDP leads to a 0.683% rise in transaction measures; for the exporting country, a 1% increase in GDP results in a 0.721% increase in transaction measures; a 1% increase in geographical distance decreases transaction measures by 0.547%; and a 1% increase in tariff rates reduces transaction measures by 1.235%, indicating that tariffs remain a significant barrier.

(3) Machine learning application: A measure-perception graph neural network was constructed to force the measure of zero-measure subsets to be 0 through completeness regularization. The model measure estimation MSE reached 9.8×10^5, and MAE was 765.32 yuan, significantly better than the traditional model, which verified the effectiveness of the measure theory in improving the accuracy of the algorithm.

In summary, this study establishes a comprehensive measurement system for the cross-border e-commerce market using the Kardiaorodis expansion theorem. Theoretically, it achieves rigorous mathematical derivation from basic space to measurement completeness, filling the gap in the application of mathematical theory in the cross-border e-commerce sector. Empirically, it verifies the adaptability and practical value of the measurement space through multi-dimensional analysis, providing a mathematical tool that is both theoretically rigorous and practically applicable for market quantitative analysis, dynamic modeling, and risk assessment.

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