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Modeling and Numerical Solution of Optimal Investment and Reinsurance Problems in Ambiguity Markets

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Abstract: In a framework of market incompleteness induced by ambiguity, this paper employs stochastic processes and stochastic analysis to formulate the decision-making problem concerning investment, consumption, and proportional reinsurance in an ambiguous market as a one-dimensional stochastic optimal control problem over a finite time horizon. Specifically, the insurer aims to maximize the utility of terminal wealth through dynamic optimal strategies, which is inherently a forward – backward stochastic differential equation system. The ambiguity in the model is captured by the Chen – Epstein multiple-priors framework, leading to a fully nonlinear Hamilton – Jacobi – Bellman equation that is generally analytically intractable. To address this, an implicit finite difference scheme is designed to numerically solve the value function as well as the optimal investment proportion, consumption strategy, and retention ratio. Furthermore, a systematic analysis is conducted to examine the quantitative impact of key market parameter variations on these optimal strategies. The findings provide theoretical support and numerical decision-making references for insurance institutions in asset allocation and risk management under complex and uncertain environments.

Keywords: reinsurance; ambiguity; stochastic differential utility; finite difference method; HJB equation

1. Introduction

The problem of investment, consumption and reinsurance is a special kind of portfolio problem, which is paid attention by domestic and international scholars. Since the reinsurance is an effective approach for risk-management, it is of practical importance to derive the optimal strategies.

In this paper, we focus on an investment, consumption and reinsurance problem in ambiguity markets. Our objective is to explore the optimal strategies to maximize the utility function of consumption and terminal wealth for the insurer. A significant amount of portfolio problems have been studied under different market assumptions. The foundational framework for continuous-time portfolio selection was established by [1], whose application of stochastic control theory yielded landmark analytical solutions. After the work of Merton, different generalizations of this classical model have been studied by different directions and different techniques. For instance, refs. [2,3] studied the optimal investment problem of minimizing the ruin probability under different conditions. Besides, consideration of the reinsurance is another natural extension to the classic investment and consumption framework [4].

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Schmidli considered dynamic proportional reinsurance strategy and studied the minimum ruin probability under the classical risk model and the diffusion risk model respectively [5]. On the other hand, some scholars also consider this kind of problems based on the maximization of utility. For example, Cao and Wan applied the stochastic technique to study the investment and proportional reinsurance problem based on maximizing the utility and get the explicit solution [6].

The above results were almost achieved under the assumption of complete markets. In reality, there are many uncertainties and diversities in financial markets, which make the investment theory become increasingly complicated. However, in order to make the results more realistic, it is significant to explore the investment problem in incomplete markets.

Actually, the utility maximization problem in ambiguity markets has been attracted considerable research attention recently. Chen and Epstein proposed the multiple-priors utility model to discuss the ambiguity problem [7]. It is assumed that the investor does not know the probability measure of the market, rank the uncertain prospects according to a multiple-priors model, and chose a worst case from a family continuous probability measures. Based on the robust utility theory, there are many specific problems are solved. For a consumption-investment problem, Schied focused on maximizing the utility of both terminal wealth and intertemporal consumption by duality theory under model uncertainty and obtain it’s explicit formulas for the optimal strategy and value function [8]. Similar problems can be referred to the paper of [9]. In addition, it is also important to study the investment and reinsurance problem in ambiguity markets. Zhang and Siu formulated an reinsurance and investment problem as two-player, zero-sum, stochastic differential games between the insurer and the market under model uncertainty [10]. They derived the closed-form solution to maximize the expected utility or minimize the discounted penalty of ruin. Lin et al. also used the game-theory approach to discuss an optimal portfolio selection problem for the insurer in a jump-diffusion model under model uncertainty [11]. They only considered an exponential utility of terminal surplus, and obtained the closed-form solutions in both the jump-diffusion risk process and its diffusion approximation.

In our paper, we study an investment, consumption and reinsurance problem by stochastic control theory and dynamic programming principle in ambiguity market. However, different from the previous literatures, the utility function of the insurer is formulated as an additive of consumption and terminal wealth, which is more sophisticated and difficult to derive it’s explicit solution in ambiguity markets.

Due to the inherent complexity of the fully nonlinear HJB equations arising from model ambiguity, analytical closed-form solutions are seldom attainable, necessitating the implementation of robust numerical algorithms. Boulbrachene applied finite element method to discuss the numerical solution for Hamilton-Jacobi-Bellman equations [12]. Kushner and Dupuis used the finite difference method for stochastic control problems in continuous time [13]. In fact, many practices show that the finite difference method is a better one, thus, we apply the finite difference method in this paper.

The significant innovation distinguishing our paper from the others is that we apply a multiple-priors model to explore the investment, consumption and reinsurance problem in ambiguity markets. As for the utility function of the insurer, we assume that the insurer concerned with not only the terminal wealth but also the consumption. The general idea of this paper is that we formulate the problem as an optimal stochastic control problem of forward-backward stochastic differential equation (FBSDE) via the stochastic control theory, and apply the dynamic programming principle to obtain the HJB equation for the value function, which is a fully nonlinear second-order partial differential equation.

It is difficult to obtain its analytical solution due to its complexity. However, we succeed in obtaining the numerical solution by the finite difference method and providing economic explanations to analyse the effects of market parameters on the value function and the optimal strategies. In fact, a numerical solution is also practical in financial markets. In this paper, we overcome two difficulties. Firstly, we formulate a complicated problem as a FBSDE by stochastic control theory. Secondly, our numerical results converge to the partial differential equation. It is a challenge to compute a fully nonlinear second-order partial differential equation, but we do it.

The rest of this paper is organized as follows. In Section 2, we formulate the problem. In Section 3, we apply the finite difference method to obtain the numerical results for the value function and the optimal strategies. In Section 4, we provide analysis for the effects of the market parameters on the value function and the optimal strategies. In Section 5, we investigate the effects form different level of ambiguity. Finally we conclude this paper in Section 6.

2. Formulation of the Model

In general, consider a finite investment duration [0, T] defined on a complete filtered probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, where the filtration $\mathcal{F}_t$ encapsulates all market information available up to time t, $\mathbb{P}$ is described as the risk-neutral probability measure. In reality, the financial market is incomplete, the insurer is uncertain about the market, and rank the uncertain prospects according to a multiple-priors model, which was initially introduced by [7]. The probability space set Θ is formulated as follows

$$\Theta \triangleq \left\{ \mathbb{Q}: \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T |\xi_s|^2 ds - \int_0^T \xi_s dW_s\right) \right\},$$

for $\xi \in \Xi$, where Ξ is a set of $\mathcal{F}_t$ adaptive stochastic process with a values on a range $\mathcal{C}=[-\underline{k}, \bar{k}]$, and $-\underline{k}, \bar{k}$ are non-negative constants, besides, W is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$.

2.1. The Surplus Process and Proportional Reinsurance

Assume that $\widehat{R}_t$ stands for the surplus process. From the classical risk model, the surplus process could be governed by the following stochastic differential equations.

$$d\widehat{R}_t = a dt - dQ_t \tag{1}$$

where $t \geqslant 0$, a is a constant insurance premium rate, and $Q_t = \left(\sum_{i=1}^{N_t} Y_i\right)$. It is worth mentioning that N_t follows a Poisson process with a constant intensity λ, known as the number of claims in the time interval [0, t]. The claim sizes $\{Y_i,\ i \geqslant 1\}$ are independent and identically distributed non-negative random variables with $\mathrm{E}_{\mathbb{P}}\left(Y_i\right) = \mu_1, \mathrm{E}_{\mathbb{P}}\left(Y_i^2\right) = \mu_2$. N_t and Y_i are adapted to a filtered probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$. In addition, N_t, Y_i and W_t are independent with each other. According to the premium principle, a positive constant safety loading η of insurer will be set to satisfy $a = (1+\eta)\lambda\mu_1$.

From the view of [14], the dQ_t can be approached by the diffusion model

$$dQ_t = \lambda\mu_1 dt - \sqrt{\lambda\mu_2}\, dW_t^1 \tag{2}$$

where W_t^1 is a standard Brownian motion and independent with W. Without loss of generalization, we suppose $\{\mathcal{F}_t\}_{0\le t\le T}$ is a complete natural filtration generated by W and W_t^1. In order to avoid catastrophe risk, the insurer will consider to purchase proportional reinsurance to transfer part of potential risk. Suppose that the self-retention proportion of the insurer is $q_t \in [0, 1]$ and $1 - q_t$ is the reinsurance proportion. In order to purchase this reinsurance contract, the insurer has to pay the premium at a rate of $a_1 = (1+\theta)\left(1-q_t\right)\lambda\mu_1$, where $1 \ge \theta > \eta \ge 0$ is the safety loading of the reinsurance business.

Referring to the paper of [5], the surplus process of the insurer after purchasing the proportional reinsurance is

$$\begin{aligned} dR_t &= \left[(1+\eta)\lambda\mu_1 - (1+\theta)\left(1-q_t\right)\lambda\mu_1\right]dt - \left(\lambda\mu_1 q_t dt - \sqrt{\lambda\mu_2}\, q_t dW_t^1\right) \\ &= \lambda\mu_1\left(\theta q_t + \eta - \theta\right)dt + \sqrt{\lambda\mu_2}\, q_t dW_t^1. \end{aligned} \tag{3}$$

Remark: There are three parts in surplus process dR_t, including premium income adt, reinsurance premium expenses a_1dt, and the claim cost $\left(\lambda\mu_1 q_t dt - \sqrt{\lambda\mu_2}\, q_t dW_t^1\right)$.

2.2. The Financial Investment Market

Suppose that the financial investment market is composed of one risk-free asset and one risky asset which are traded continuously over [0, T]. The price process B_t of the risk-free

$$dB_t = r(t)B_t dt \tag{4}$$

where $r(t)$ is assumed to be a deterministic and continuous function, and represents the interest rate. The price of the risky asset is driven by a geometric Brownian motion

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dW_t, \tag{5}$$

where $\mu(t) > 0$ and $\sigma(t) > 0$ are continuous and deterministic functions, known as the expected rate of return and volatility of the risky asset, respectively.

2.3. The Insurer' s Wealth Process

Let X_t be the total wealth of the insurer at time $t \in [0, \mathrm{T}]$, π_t and $X_t - \pi_t$ are the amount of money invested in the risky asset and risk-free asset respectively. In addition, C_t indicates the consumption rate. Ignore the transaction cost, the dynamic of X_t is given by

$$\begin{aligned} dX_t &= \frac{X_t - \pi_t}{B_t} dB_t + \frac{\pi_t}{S_t} dS_t + dR_t - C_t dt \\ &= \frac{X_t - \pi_t}{B_t} r(t) B_t dt + \frac{\pi_t}{S_t}\big(\mu(t) S_t dt + \sigma(t) S_t dW_t\big) + \lambda\mu_1\big(\theta q_t + \eta - \theta\big) dt + \sqrt{\lambda\mu_2}\, q_t dW_t^1 - C_t dt \\ &= \big[r(t) X_t + (\mu(t) - r(t))\pi_t + \lambda\mu_1 \theta q_t + \lambda\mu_1(\eta - \theta) - C_t\big] dt + \sqrt{\lambda\mu_2}\, q_t dW_t^1 + \sigma(t)\pi_t dW_t \end{aligned} \tag{6}$$

Therefore, X_s is formulated as the stochastic differential equation under $\mathbb{P}$

$$\begin{aligned} X_s^{t,x;\pi,q,C} &= x + \int_t^s \big[r(u) X_u^{t,x;\pi,q,C} + (\mu(u) - r(u))\pi_u + \lambda\mu_1(\eta - \theta) - C_u\big] du \\ &+ \int_t^s \sqrt{\lambda\mu_2}\, q_u dW_u^1 + \int_t^s \sigma(u)\pi_u dW_u \end{aligned} \tag{7}$$

where x is the wealth at time t, and (π, q, C) represent the strategies of investment, reinsurance and consumption belonging to the admissible controls set

$$\mathcal{A}(t,x) \triangleq \left\{(\pi, q, C): \sup_{\mathbb{Q}} \mathrm{E}_{\mathbb{Q}}\left[\int_t^T \pi_s^2 ds\right] < \infty, -\infty < \pi < +\infty, 0 \le q \le 1, C \ge 0, X \ge 0\right\}.$$

Remark: $\mathcal{A}$ is the set of all admissible controls, $\forall \Pi = (\pi, q, C) \in \mathcal{A}, \Pi$ is progressively measurable with respect to the $\{\mathcal{F}_t\}_{0 \le t \le T}$ filtration.

In reality, the insurer makes the decision under $\mathbb{Q}$ but not $\mathbb{P}$, as he/she is uncertain about the risk-neutral probability $\mathbb{P}$. From the Girsanov theorem, the X_t can be described as the following equation under $\mathbb{Q}$.

$$\begin{aligned} X_s^{t,x;\pi,q,C} &= x + \int_t^s \big[r(u) X_u^{t,x;\pi,q,C} + (\mu(u) - r(u))\pi_u + \lambda\mu_1(\eta - \theta) - C_u \\ &- \sigma(u)\pi_u \xi_u\big] du + \int_t^s \sqrt{\lambda\mu_2}\, q_u dW_u^1 + \int_t^s \sigma(u)\pi_u d\tilde{W}_u \end{aligned} \tag{8}$$

where $\tilde{W}_t = W_t + \int_0^t \xi_u du$ is the stochastic Brownian motion under $\mathbb{Q}$.

2.4. Optimization of the Model and It' s HJB Equation

In this paper, our aim is to explore the optimal strategies to maximize the expected utility of consumption and terminal wealth in the finite time horizon. Define the bankruptcy time as follows

$$\tau \triangleq \inf\left\{s: X_S < 0\right\}.$$

As a balance between average return and risk, the utility function is usually used to describe the satisfaction from investment and consumption. The insurer has an additive utility function defined on the priors set Θ, which is formulated as a stochastic differential utility in [15], and it can be represented as the following formula under measure $\mathbb{Q}$.

$$U_t^{\mathbb{Q}} \triangleq \mathrm{E}_{\mathbb{Q}}\left[\int_t^{T\wedge\tau} e^{-r(s-t)}\big(1 - e^{-bC_s}\big) ds + e^{-(T\wedge\tau - t)} \frac{\big(X_{T\wedge\tau}^{t,x;\pi,q,C}\big)^p}{p}\right], \forall t \in [0, T]$$

Remark: U_t is assumed to be composed of the terminal wealth and the exponential utility consumption function, where C is the consumption rate, b is a positive absolute risk aversion coefficient. In addition, suppose that $p \in (0, 1)$, and $\frac{X^p}{p}$ is a power utility function of wealth, where $1 - p$ stands for the coefficient of relative risk aversion.

As there exists ambiguity about the risk-neutral probability measure $\mathbb{P}$, the insurer is uncertain about the measurement of the markets. According to the method of literature [7], the insurer will choose the worst case of the stochastic differential utility from the priors set Θ:

$$U_t = \inf_{\mathbb{Q} \in \Theta} U_t^{\mathbb{Q}}$$

Hence, the U_t can be represented as the following BSDE:

$$\begin{aligned} U_t = {} & \frac{\left(X_{T \wedge \tau}^{t,x;\pi,q,C} \right)^p}{p} + \int_t^{T \wedge \tau} \left[\left(1 - e^{-bC_u} \right) - r(u) U_u - \bar{k} max\left(Z_u, 0 \right) - \underline{k}\, max\left(-Z_u, 0 \right) \right] du \\ & - \int_t^{T \wedge \tau} Z_u^1 dW_u^1 - \int_t^{T \wedge \tau} Z_u dW_u, \forall t \in [0, T] \end{aligned} \tag{9}$$

In this paper, the insurer's objective is to maximize the utility function by choosing a strategy $(\pi, q, C) \in \mathcal{A}(t,x)$. Let $V(t,x)$ represent the value function of the stochastic control problem, which is equivalent to the optimal objective function $U\left(t, x; \pi^*, q^*, C^*\right)$, and the $U(t,x;\pi,q,C)$ is the utility function from Formula (9). Thus, we have the following equality:

$$V(t,x) \triangleq U\left(t, x; \pi^*, q^*, C^*\right) = \sup_{(\pi,q,C) \in \mathcal{A}(t,x)} U(t,x;\pi,q,C)$$

From the stochastic control theory of [16], the value function is the viscosity solution of the corresponding HJB equation:

$$\begin{cases} \partial_t V + \sup\limits_{\substack{0 \le q \le 1, C \ge 0 \\ -\infty < \pi < +\infty}} \left\{ \frac{1}{2} \left(\lambda \mu_2 q^2 + \sigma^2 \pi^2 \right) \partial_{xx} V + \left[rx + (\mu - r)\pi + \lambda \mu_1 \theta q + \lambda \mu_1 (\eta - \theta) - C \right] \partial_x V \right. \\ \qquad \left. + 1 - e^{-bC} - rV - \bar{k} \max\left(\sigma \pi \partial_x V, 0 \right) - \underline{k} \max\left(-\sigma \pi \partial_x V, 0 \right) \right\} = 0, & (t,x) \in [0,T] \times (0,\infty), \\ V(T,x) = \dfrac{x^p}{p}, & x \in [0,\infty), \\ V(t,0) = 0, & t \in [0,T). \end{cases} \tag{10}$$

3. The Finite Difference Method to Obtain the Numerical Result

We have obtained the HJB equation for the value function in the previous section. As we known, it is difficult to get the explicit solution to this problem due to the HJB equation is a fully nonlinear second-order PDE. In this section, an implicit difference method is used to deal with the HJB equation. We obtain the numerical result for the value function and the optimal strategies.

3.1. Discuss to the Optimal Strategies

The equivalent PDE of Equation (10) could be written as:

$$\begin{cases} \partial_t V + \left\{ \frac{1}{2} \left(\lambda \mu_2 q^{*2} + \sigma^2 \pi^{*2} \right) \partial_{xx} V + \left[rx + (\mu - r)\pi^* + \lambda \mu_1 \theta q^* + \lambda \mu_1 (\eta - \theta) - C^* \right] \partial_x V \right. \\ \left. + 1 - e^{-bC} - rV - \bar{k} \max\left(\sigma \pi \partial_x V, 0 \right) - \underline{k} \max\left(-\sigma \pi \partial_x V, 0 \right) \right\} = 0, (t,x) \in [0,T] \times (0,\infty), \\ V(T,x) = \dfrac{x^p}{p}, \qquad x \in [0,\infty), \\ V(t,0) = 0, \qquad t \in [0,T). \end{cases} \tag{11}$$

where,

$$\begin{cases} \pi^* = \operatorname*{argsup}\limits_{-\infty < \pi < +\infty} \left\{ \frac{1}{2} \sigma^2 \partial_{xx} V \pi^2 + (\mu - r) \partial_x V \pi - \bar{k} \max\left(\sigma \pi \partial_x V, 0 \right) - \underline{k} \max\left(-\sigma \pi \partial_x V, 0 \right) \right\}, \\ q^* = \operatorname*{argsup}\limits_{0 \le q \le 1} \left\{ \frac{1}{2} \lambda \mu_2 \partial_{xx} V q^2 + \lambda \mu_1 \theta \partial_x V q \right\}, \\ C^* = \operatorname*{argsup}\limits_{C \ge 0} \left\{ -C \partial_x V - e^{-bC} \right\}. \end{cases} \tag{12}$$

With the help of the indicator functions, the optimal strategies can be obtained from Equation (12)

$$\begin{cases} \pi^*=-\dfrac{(\mu-r-\bar{k}\sigma)\partial_x V}{\sigma^2\partial_{xx}V}I_1-\dfrac{(\mu-r+\underline{k}\,\sigma)\partial_x V}{\sigma^2\partial_{xx}V}I_2, I_1=I_{\mu-r-\bar{k}\sigma>0}, I_2=I_{\mu-r+\underline{k}\,\sigma<0}, \\ q^*=-\dfrac{\mu_1\theta\partial_x V}{\mu_2\partial_{xx}V}I_3+I_4, I_3=I_{\mu_1\theta\partial_x V\le-\mu_2\partial_{xx}V}, I_4=I_{\mu_1\theta\partial_x V>-\mu_2\partial_{xx}V} \\ C^*=-\dfrac{1}{b}\ln\dfrac{\partial_x V}{b}I_5, I_5=I_{\partial_x V<b}. \end{cases} \tag{13}$$

3.2. The Idea of Finite Difference Method

To approximate the solution of the governing PDEs, we employ the finite difference method, which discretizes the continuous differential operators into a system of algebraic equations over a predefined grid. Give a computational region $\{0\le x\le M, 0\le t\le T\}$, and we discrete x and t to obtain a size of $n \times m$ grid points. Let $dx=\frac{M}{n}, dt=\frac{T}{m}$ represent the grid spacings, hence, $(i,j)=((i-1)dt,(j-1)dx)$, where $i=1,2,\cdots,m+1; j=1,2,\cdots,n+1$. Clearly, each grid point (i,j) corresponds to $V_{i,j}, \partial_t V_{i,j}, \partial_x V_{i,j}, \partial_{xx}V_{i,j}$.

First of all, we use explicit difference method to discrete the Formula (13).

$$\begin{cases} \pi^*_{i,j}=-\dfrac{(\mu-r-\bar{k}\sigma)\partial_x V_{i,j}}{\sigma^2\partial_{xx}V_{i,j}}I_1-\dfrac{(\mu-r+\underline{k}\,\sigma)\partial_x V_{i,j}}{\sigma^2\partial_{xx}V_{i,j}}I_2 \\ q^*_{i,j}=-\dfrac{\mu_1\theta\partial_x V_{i,j}}{\mu_2\partial_{xx}V_{i,j}}I_{3,i,j}+I_{4,i,j} \\ C^*_{i,j}=-\dfrac{1}{b}\ln\dfrac{\partial_x V_{i,j}}{b}I_{5,i,j}. \end{cases} \tag{14}$$

where

$$\begin{cases} \partial_x V_{i,j}=\dfrac{V_{i,j+1}-V_{i,j-1}}{2dx}, \\ \partial_{xx}V_{i,j}=\dfrac{V_{i,j+1}+V_{i,j-1}-2V_{i,j}}{(dx)^2}, \\ I_{3,i,j}=I_{\left(\mu_1\theta\partial_x V_{i,j}\le-\mu_2\partial_{xx}V_{i,j}\right)}, \\ I_{4,i,j}=I_{\left(\mu_1\theta\partial_x V_{i,j}>-\mu_2\partial_{xx}V_{i,j}\right)}, \\ I_{5,i,j}=I_{\left(\partial_x V_{i,j}<b\right)}. \end{cases}$$

Next, we apply implicit difference scheme and upwind technique to discrete the PDE. Beforehand, we need to add the boundary condition $V(t, M)$. As a matter of fact, we can let $V(t,M)=\frac{M^p}{p}$ $(t\in[0,T])$,which can be compatible with the terminal condition $V(T,x)=\frac{x^p}{p}$ at point (T, M). The more details about this idea, please refer to [17].

$$\begin{cases} \dfrac{V_{i+1,j}-V^{k+1}_{i,j}}{dt}+\beta^k_{1,i,j}\dfrac{V^{k+1}_{i,j+1}+V^{k+1}_{i,j-1}-2V^{k+1}_{i,j}}{(dx)^2}+\left(\beta^k_{2,i,j}\right)^+\dfrac{V^{k+1}_{i,j+1}-V^{k+1}_{i,j}}{dx}-\left(\beta^k_{2,i,j}\right)^-\dfrac{V^{k+1}_{i,j}-V^{k+1}_{i,j-1}}{dx} \\ -rV^{k+1}_{i,j}=-1+e^{-bC^{*k}_{i+1,j}}, \\ V_{m+1,j}=\dfrac{x_j^p}{p}, j=2,\cdots,n, \\ V_{i,1}=0, i=m,m-1,\cdots,1, \\ V_{i,n+1}=\dfrac{M^p}{p}, i=m,m-1,\cdots,1. \end{cases} \tag{15}$$

where

$$\begin{cases} \beta^k_{1,i,j}=\dfrac{1}{2}\left(\lambda\mu_2 {q^{*k}_{i+1,j}}^2+\sigma^2 {\pi^{*k}_{i+1,j}}^2\right), \\ \beta^k_{2,i,j}=rx_j+(\mu-r)\pi^{*k}_{i+1,j}+\lambda\mu_1\theta q^{*k}_{i+1,j}+\lambda\mu_1(\eta-\theta)-C^{*k}_{i+1,j}-\bar{k}max\left(\sigma\pi^{*k}_{i+1,j},0\right) \\ -\underline{k}\,max\left(-\sigma\pi^{*k}_{i+1,j},0\right). \end{cases}$$

Referring to the view of [18], we use upwind scheme to discrete $\partial_x V_{i,j}$, which is approximated by either a forward or a backward finite difference depending on the sign of its coefficient. The specific scheme is that we apply the forward difference method when $\beta^k_{2,i,j} > 0$, and oppositely, we use the backward difference method.

At last, we can obtain $\pi^*_{m+1,j}, q^*_{m+1,j}, C^*_{m+1,j}$ when we plug $V_{m+1,j}$ of the Equation (15) into the Formula (14), and get the $V_{m,j}$ when $\pi^*_{m+1,j}, q^*_{m+1,j}, C^*_{m+1,j}$ are put into the difference Equation (14). By this way, we can get

$$\pi^*_{m,j}, q^*_{m,j}, C^*_{m,j} \to V_{m-1,j} \to \cdots \to \pi^*_{2,j}, q^*_{2,j}, C^*_{2,j} \to \pi^*_{1,j}, q^*_{1,j}, C^*_{1,j}.$$

3.3. Numerical Result

According to the numerical scheme, the numerical results can be acquired. The parameters of our model are similar to the literature [19], which are given as follows in Table 1:

Table 1. parameter values.

Parameter Values				
$T = 1$	$M = 40$	$r = 0.04$	$\sigma = 0.1$	$\mu = 0.1$
$\bar{k} = \underline{k} = 0.1$	$p = 0.65$	$b = 0.3$	$\eta = 0.3$	$\theta = 0.4$
$\lambda = 3$	$\mu_1 = 0.1$	$\mu_2 = 0.2$	$m = 200$	$n = 10000$

Computing by matlab 2013a, we obtain the following results:

Figure 1 reflects the relationship of value function and wealth in finite time horizon. According to the figure, at a fix moment, the value function grows gradually as wealth level increases, but the growth speed of value function slow down with respect to the wealth level. Obviously, it comes to a conclusion that the wealth level have a positive effect on value function.

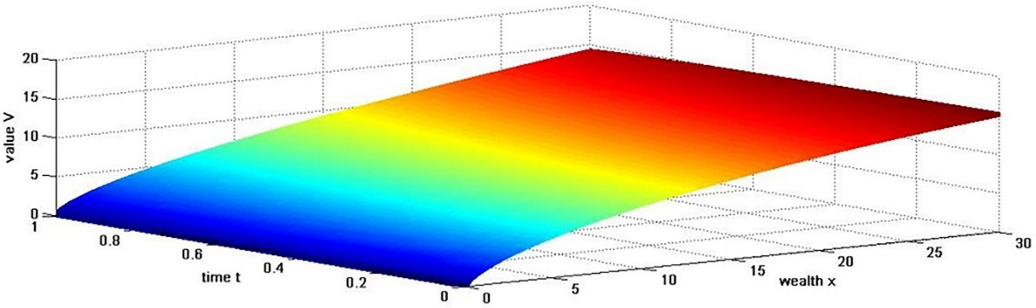

Figure 1. value function.

Figure 2 represents the change of the optimal amount of money π^* invested in risky asset with time and wealth level. We can find that at the initial time, the π^* increases gradually at a low wealth level, but it follows a slight drop as the wealth level reaches to a certain value. It is a common sense that the insurer will put more shares into risky asset to get greater benefits at a low wealth level. On the other hand, the insurer is a kind of risk avoid investor, who may keep cautious in the investment after the wealth level runs up to a certain value. However, to our surprise, the π^* shows a rapid growth with respect to the wealth level at the terminal time. It is because that the risk will reduce when time comes to the terminal.

Figure 3 shows the characteristic of the optimal self-retention proportion q^*. From the picture we can know that at any moment, the q^* rapid growth to 1 when the wealth level increases. The results is obvious, the insurer will have a good tolerance for risk when the wealth increases to a certain amount, and it will reduce the reinsurance proportion.

Figure 4 reveals the features of the optimal consumption rate C^*. Over $t \in [0, T]$, we find that the consumption rate is zero at a low wealth level and begin to increase gradually at the wealth level about 22. However, wealth level makes no difference to consumption rate at the terminal time. In reality, in order to maintain normal operation, the insurer will not consume at a low wealth level, and as the wealth accumulate to a certain amount, the insurer will increase consumption.

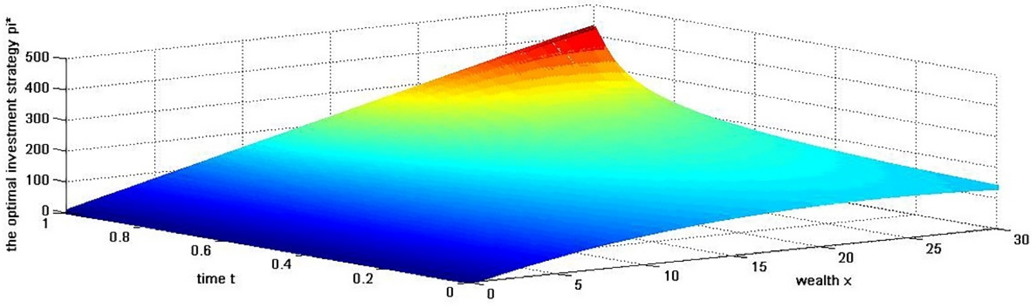

Figure 2. the optimal investment strategy.

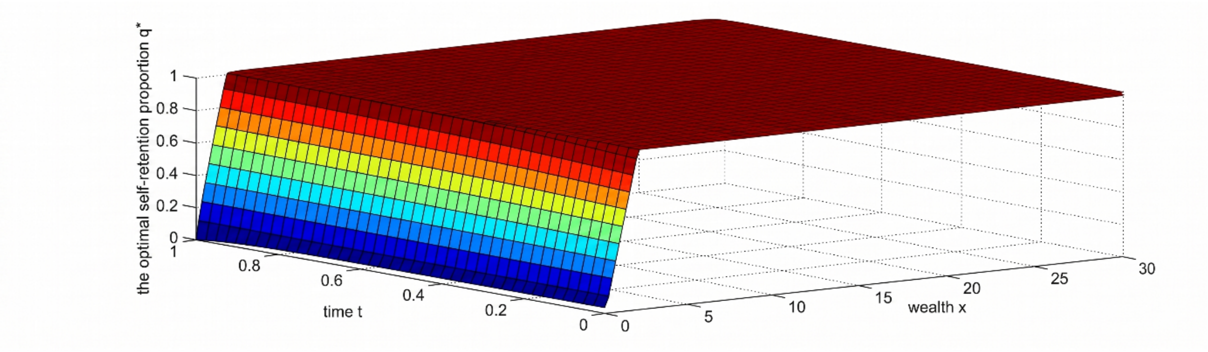

Figure 3. the optimal self-retention proportion.

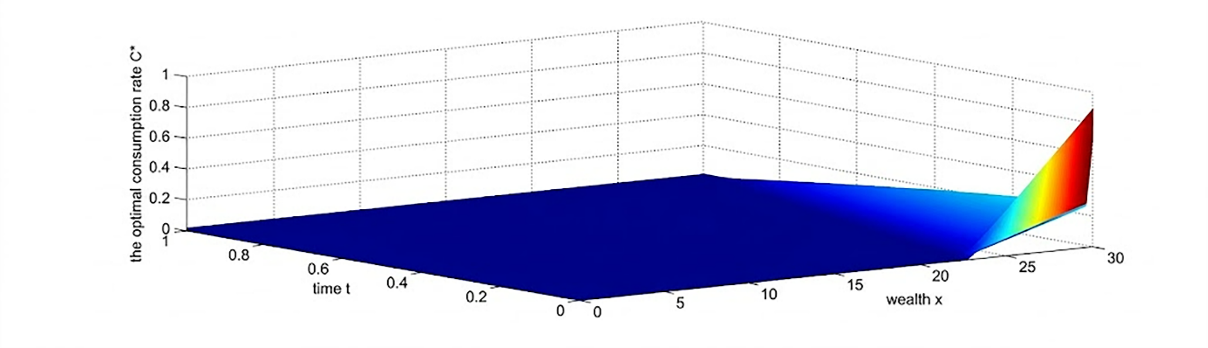

Figure 4. the optimal consumption rate.

4. The Parameters Sensitivity Analysis

In the real financial markets, many factors are constantly changing, it is necessary to analyze the influence of the parameters on the utility function and the optimal strategies. In this section, we use the method of controlling variables to analyze the effects of parameters on the utility function and the optimal strategies at initial moment.

4.1. The Influence of the Parameters on the Utility Value and Economic Analysis

From the Figures 5–8, we can analyze the effects of market parameters on the utility function and provide some economic analysis. (i) In the Figure 5, the utility function shows a slightly downward trend as $\bar{k}$ increases. In fact, a lager $\bar{k}$ leads to greater uncertainty of the market, which leads to the decrease of the utility function. (ii) The Figure 6 shows that the utility function has a general increase trend following to a smaller interest rate r. It can be attributed to the fact that a drop in interest rate means a lower cost of investment, which stimulate the investment. (iii) According to the Figure 7, the expected rate of return μ has a significant sensitivity on utility function. Obviously, the insurer will acquire more from investment return when the risky asset have a good expected rate of return. (iv) From the Figure 8, we can find that there is a negative relationship between the utility function and the volatility σ. As we know, volatility reflects the uncertainty of returns from risky asset, a higher volatility leads less money to invest in risky asset, which result in a lower utility function.

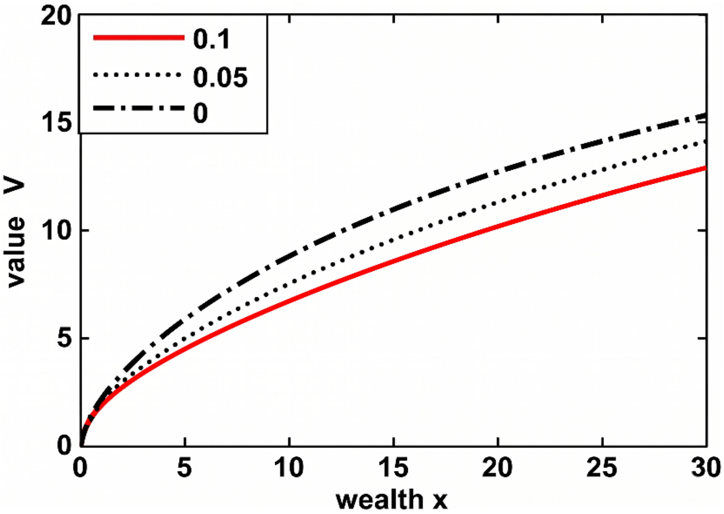

Figure 5. The impact of k on the utility value.

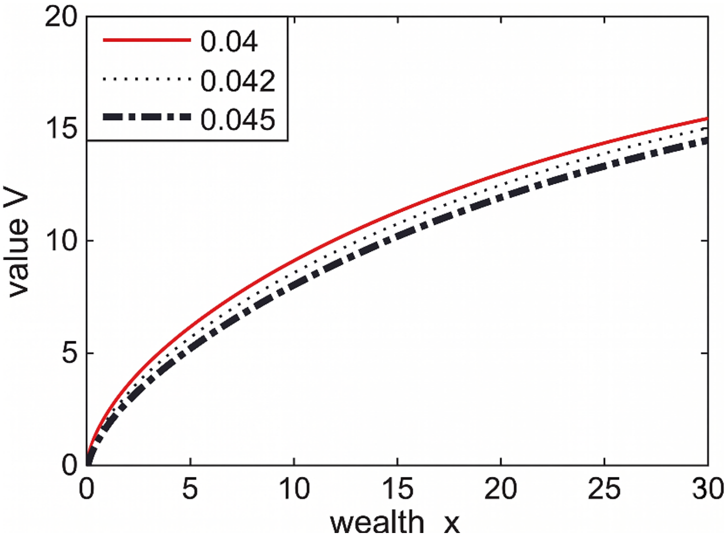

Figure 6. The impact of r on the utility value.

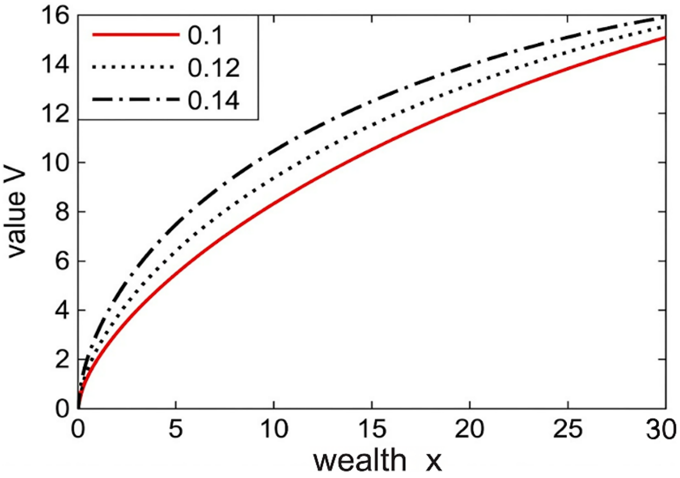

Figure 7. The impact of μ on the utility value.

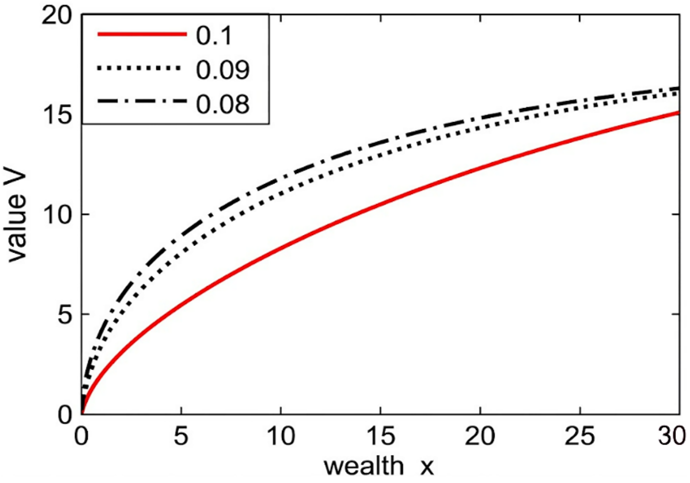

Figure 8. The impact of σ on the utility value.

4.2. The Influence of the Parameters on the Strategy π^* and Economic Analysis

As is shown in Figures 9–12, they lead to some conclusions. (i) From the Figure 9, we find that π^* appears to be decline when the k increase. In general, the amount of money invested in the risky asset will be affected by the uncertainty of the market. (ii) In the Figure 10, the interest rate r has a little sensitivity on π^*. It is obvious that the insurer will reduce the money invested in the risky asset when the interest rate raises. (iii) The Figure 11 reflects a great different impacts on π^* with different expected rate of return μ. As μ goes up, the insurer will put more money in the risky asset to obtain more revenue. (iv) Paying attention to the Figure 12, the volatility σ shows a remarkable effect on π^*, especially at a higher wealth level. It is a common sense that the insurer will reduce investment in the risky asset so as to reduce the risk at a larger volatility.

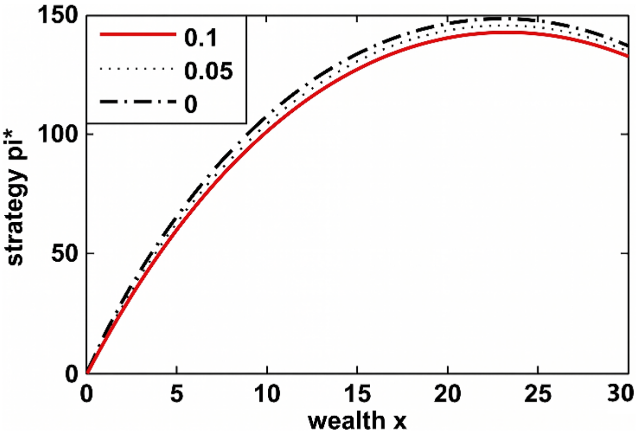

Figure 9. The impact of $\bar{k}$ on the strategy π^*.

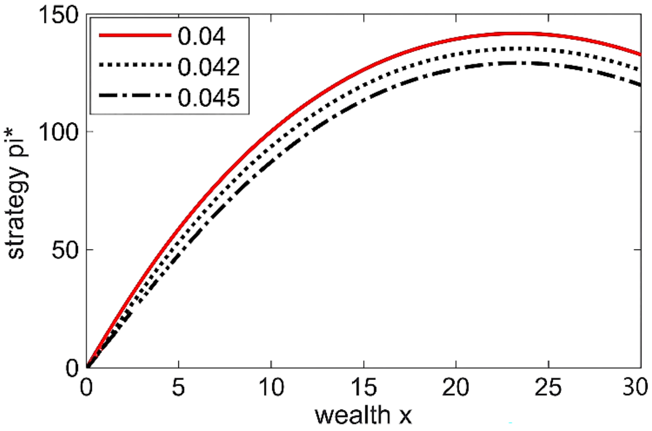

Figure 10. The impact of r on the strategy π^*.

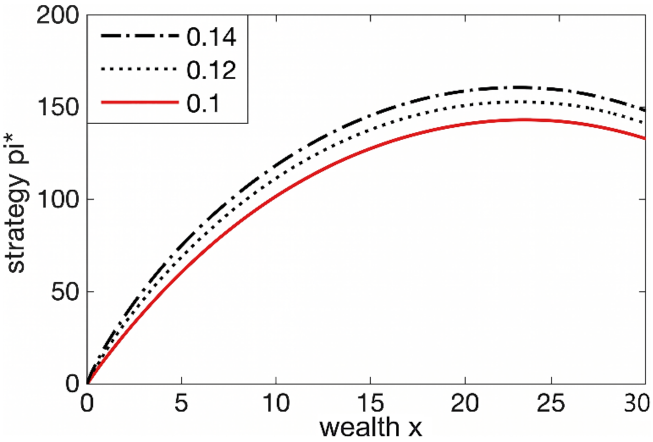

Figure 11. The impact of μ on the strategy π^*.

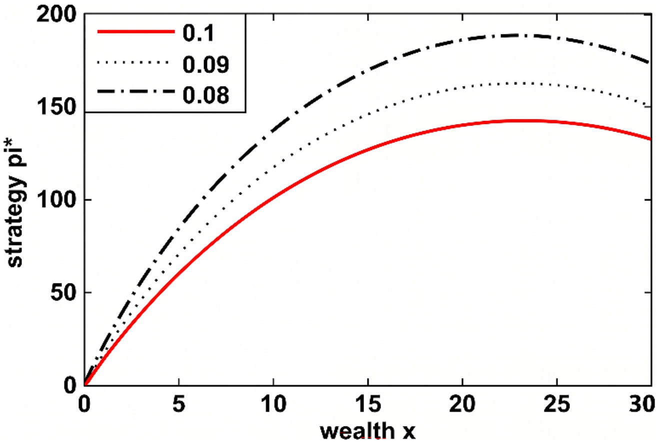

Figure 12. The impact of σ on the strategy π^*.

4.3. The Influence of the Parameters on the Strategy q^* and Economic Analysis

The Figures 13–16 illustrate the influences of the parameters on the optimal self-retention proportion q^*. (i) The Figure 13 indicates that the change of $\bar{k}$ effects q^* slightly. However, we can still find that the insurer prefers to keep a stable revenue by purchasing less reinsurance as $\bar{k}$ increases. This can be attributed to the fact that a lager $\bar{k}$ leads to less money invested in the risky asset, thus, it is not necessary for the insurer to purchase more reinsurance. (ii) From the Figure 14, it shows a trend that the interest rate r has a positive impact on q^*. It means that the insurer is stronger to endure the risk as the interest rate goes up, accordingly, it will reduce the proportion of reinsurance. (iii) In the Figure 15, the q^* towards a decrease trend as the expected rate of return μ increases. Naturally, the risk appears to be increased as more money is invested in the risky asset, which also leads to less self-retention proportion. (iv) The Figure 16 reveals that the q^* shows a slight increased when it comes to a bigger volatility σ. In this case, the insurer prefers to purchase less reinsurance to achieve the goal of profit as the money invested in the risky asset reduced.

4.4. The Influence of the Parameters on the Consumption Rate C^* and Economic Analysis

For Figures 17–20, they cover some interesting information. (i) Generally speaking, the change of k shows an effect considerably on the consumption rate C^* according to the Figure 17. The insurer will consume carefully when the $\bar{k}$ increased. (ii) In the Figure 18, the optimal consumption rate C^* appears great difference with the change of the interest rate *r*. It is a economic sense that the insurer will prefer to invest in the risk-free asset rather than consumption. (iii) From the Figure 19, we know that the optimal consumption rate C^* has a noticeable upward trend when expected rate of return μ increases. It can be explained by the fact that the insurer can get more revenue as μ increases, and has more capital to consume. (iv) In the last figure, the optimal consumption rate C^* shows a significant decline with a larger volatility σ. The insurer receives less from the risky asset when volatility increases, and the insurer will prefers to accumulate funds rather than to consume.

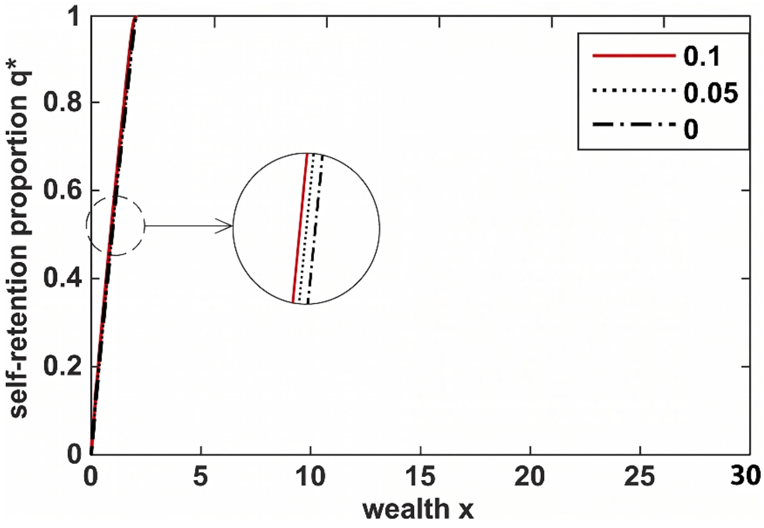

Figure 13. The impact of $\bar{k}$ on the strategy q^*.

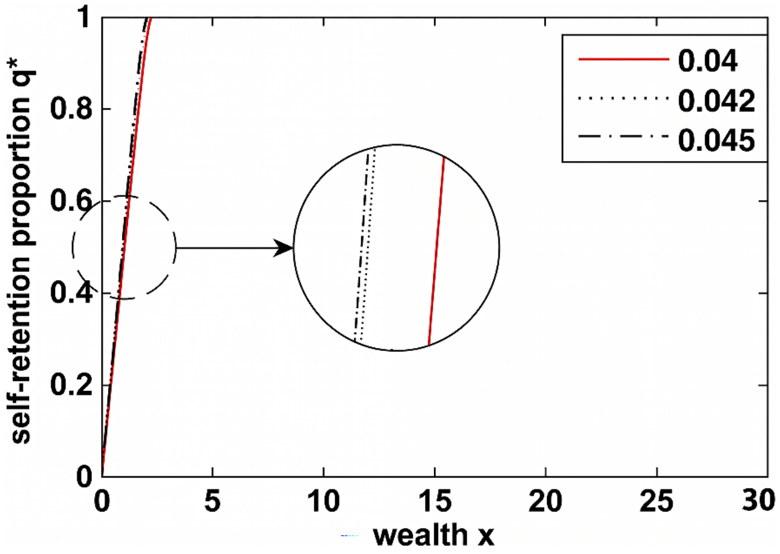

Figure 14. The impact of r on the strategy q^*.

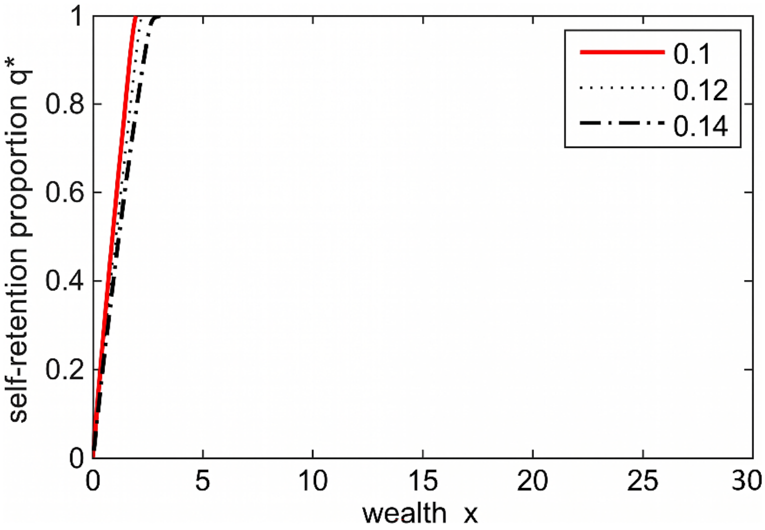

Figure 15. The impact of μ on the strategy q^*.

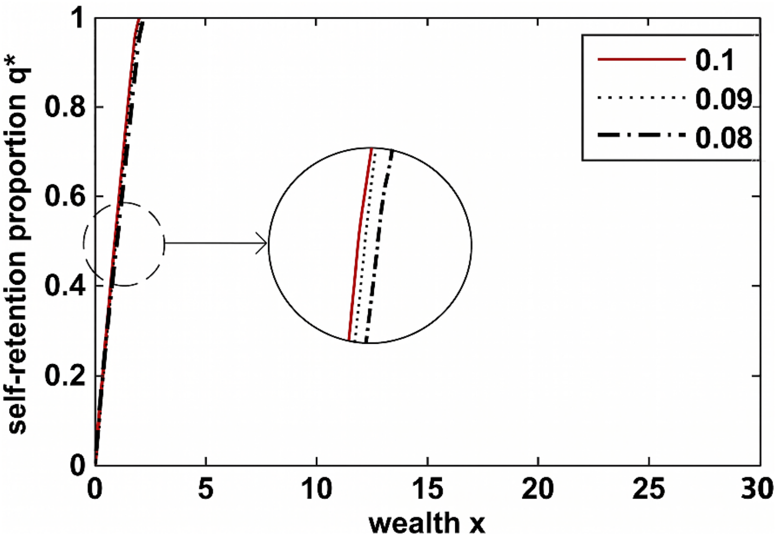

Figure 16. The impact of σ on the strategy q^*.

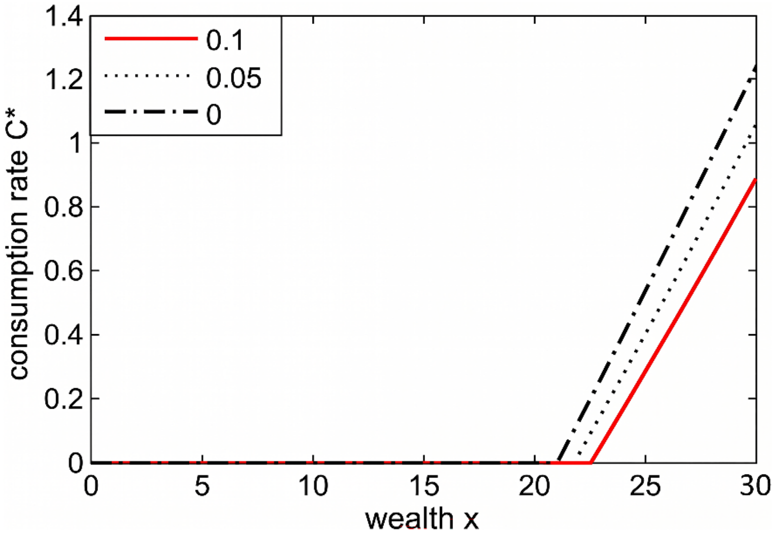

Figure 17. The impact of $\bar{k}$ on the strategy C^*.

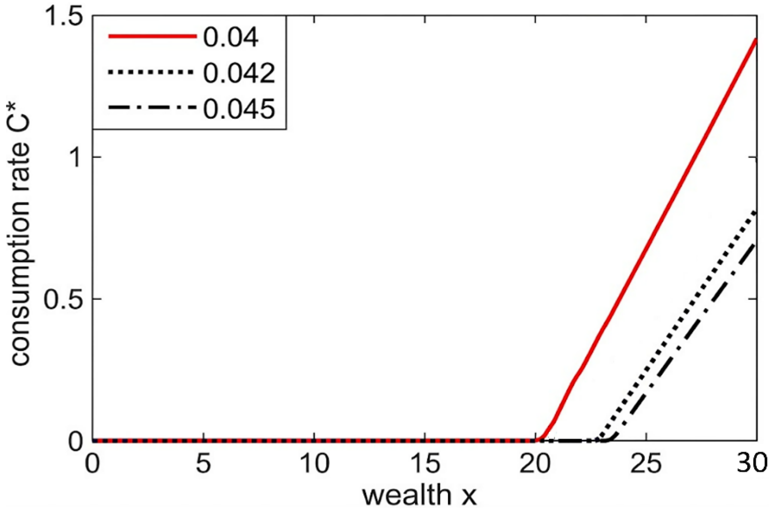

Figure 18. The impact of r on the strategy C^*.

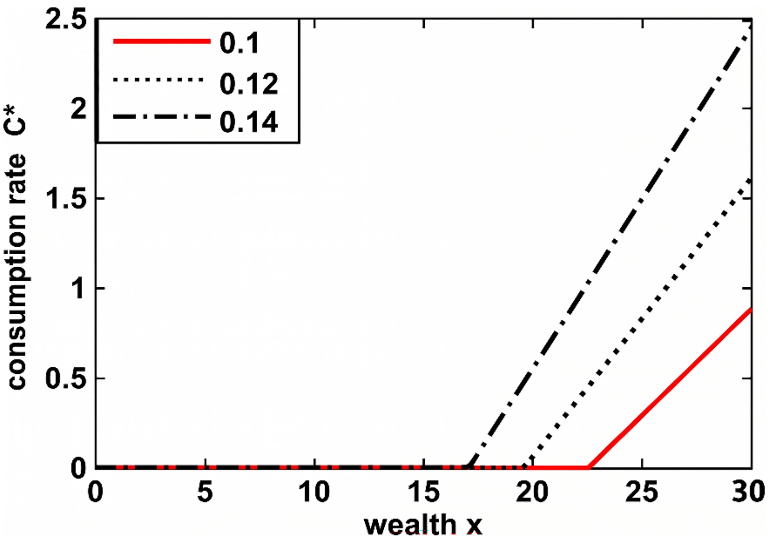

Figure 19. The impact of μ on the strategy C^*.

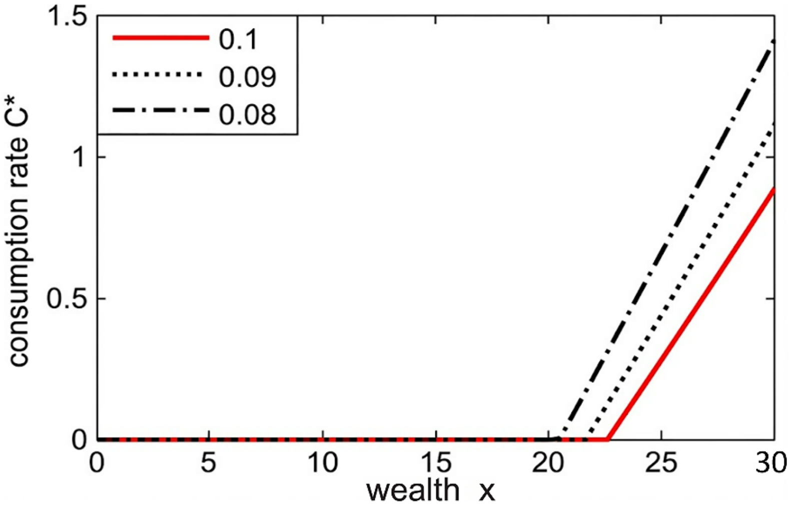

Figure 20. The impact of σ on the strategy C^*.

5. Effects from Different Levels of Ambiguity

In this section, we explore the impact of the insurer from different levels of ambiguity. Compute $\Delta\bar{V}_0 = V_0^{\bar{k}} - V_0, \Delta\bar{\pi}_0 = \pi_0^{\bar{k}} - \pi_0, \Delta\bar{q}_0 = q_0^{\bar{k}} - q_0, \Delta\bar{C}_0 = C_0^{\bar{k}},$ which show the average changes of V, π, q and C comparing to $\bar{k} = 0$ at different $\bar{k}$. The results are as follows in Table 2:

Table 2. The changes of the utility and strategies.

$\bar{k}$	$\Delta\bar{V}_0$	$\Delta\bar{\pi}_0$	$\Delta\bar{q}_0$	$\Delta\bar{C}_0$
0.1	−0.3369	−4.826	0.0041	−0.1711
0.2	−0.6609	−11.1612	0.0066	−0.3203
0.3	−0.9611	−19.9260	0.0076	−0.4465
0.4	−1.2207	−32.5946	0.0077	−0.5481
0.5	−1.4138	−50.0773	0.0077	−0.6216

From the table, we can find that comparing to the $\bar{k} = 0$, the average loss of utility will be affected more by a greater $\bar{k}$, every increase of $\bar{k}$ by 0.1, the average loss of utility will increase about 0.3. As for the strategies, the effects of $\bar{k}$ on π and C are significant, but it is not obvious for q. The changes of V, π, q and C will converge to 0 as $\bar{k}$ tends to 0.

6. Conclusions

In this paper, we apply the assumption of multiple-priors model to study the optimal investment, consumption and reinsurance problem in an ambiguity market. Considering the ambiguity of the market is one of the innovations in this paper. With the stochastic control theory, we obtain HJB equation for the value function, which is a fully nonlinear second-order PDE. We solve it via the finite difference method and achieve it's numerical solution. At last, we provide the economic interpretations to illustrate the effects of parameters on the value function and the optimal strategies. The results of this paper have a certain guiding for decision-making of the insurer.

In the future research, we can consider the stochastic interest rate model. It will lead to some sophisticated models, and we leave these problems for future research.

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Author Contributions

Writing—original draft, X.L., X.Z., D.W. and R.M.; writing—review and editing, X.L., X.Z., D.W. and R.M. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

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