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Article

Understanding Ramsey-de Finetti Probabilities and the St. Petersburg Paradox

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Abstract: We provide simple interpretations of probability with a view towards its potential for real world (i.e., practical) applications. Our discussion is along the lines initiated by David Ramsey and Bruno de Finetti, though we do not provide a theory of probability but merely use their axioms and results to interpret mathematical probability. Subsequently, we provide a critique as well as a resolution of the well-known St. Petersburg paradox that is consistent with expected utility theory.

Keywords: Ramsey-de Finetti probability; sure gain; conditional probability; St. Petersburg paradox

1. Introduction

Our purpose here is to provide simple interpretations of probability with a view towards its potential for real world (i.e., practical) applications. Our discussion is along the lines initiated by David Ramsey [1,2] and Bruno de Finetti [3–5]. We hope that this will be easier to understand than section 3 of [6]. Unlike David Ramsey and Bruno de Finetti, we do not provide a theory of probability; we merely use their axioms and results to interpret mathematical probability. An interesting discussion of the interpretation of probability that concerns us here is available in [7]. Critiques of the probability theory due to de Finetti and Ramsey are advised to read the paper by Velupillai [8]. Subsequently, we provide a critique as well as a resolution of the well-known St. Petersburg paradox (see section 2.4 of [9] that is consistent with expected utility theory, the latter being understood as discussed in section 4 of [6]. Here we view the paradox differently from the way it is discussed in [6]. Our contention here as well as in [6] is that the St. Petersburg paradox is an example of the human mind processing information about uncertainty differently from the way probability theory would expect us to do so, rather than a reason for expected utility theory.

In what follows we assume familiarity with the concept of mathematical probability at a very basic level, e. g., [10] available at:https://www.sfu.ca/~mdevos/notes/graph/probprimer.pdf.

2. Interpretation of Probability and Conditional Probability

Given an event E, a simple bet based on E, is a lottery ticket that returns 1 unit of money if E occurs and nothing otherwise.

According to the "Ramsey-de Finetti interpretation of mathematical (Kolmogrov) probability", the subjective probability of an event E and the price for a "simple bet based on E" perceived by an individual in a

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thought experiment, are such that the expected returns from such a bet is zero.

It is worth noting that for a price ‘p’ for a “simple bet based on E” there exists a probability of E such that the expected returns from buying a “simple bet based on E” is zero if and only if the following two conditions are satisfied: (a) there is no “sure gain” for a buyer of the simple bet, i.e., either $1-p \leqslant 0$ or $-p \leqslant 0$; and (b) there is no “sure gain” for the seller of the bet, i.e., either $p - 1 \leqslant 0$ or $p \leqslant 0$.

For if both 1-p and -p are strictly positive or both (p-1) and p are strictly positive (i.e., both 1-p and p are strictly negative), then there does not exist $\alpha \in [0,1]$ such that $\alpha(1-p) - (1-\alpha)p = 0$. Thus, if there exists $\alpha \in [0,1]$ such that $\alpha(1-p) - (1-\alpha)p = 0$, then it must be the case that both (a) and (b) are satisfied.

The absence of a “sure gain” for a buyer of a “simple bet based on E” is equivalent to the requirement $p \geqslant 0$. The absence of a “sure gain” for a seller of a “simple bet based on E” is equivalent to the requirement $p \leqslant 1$. Thus, the absence of a “sure gain” for a buyer in conjunction with the absence of a “sure gain” for a seller is equivalent to the requirement that $p \in [0,1]$, which implies that for $\alpha = p \in [0,1]$, $\alpha(1-p) - (1-\alpha)p = \alpha - p = 0$.

Thus, in the context of the thought experiment discussed above, if P(E) is the “subjective probability” of the event E, and p is the price of a “simple bet based on E”, then it must be the case that $p \in [0,1]$, thereby implying $p = P(E)$.

If p(E) (> P(E)) is the price of a “simple bet based on E”, then the expected returns from buying a “simple bet on E” would be negative.

If p(E) (< P(E)) is the price of a “simple bet based on E”, then the expected returns from selling a “simple bet on E” would be negative.

Note: For those familiar with expected utility of monetary gains and losses, it may be worth noting that the function $p \mapsto P(E)u(1-p) + (1-P(E))u(-p)$ is strictly decreasing in p, where u is a strictly increasing function of monetary gains and losses, satisfying u(0). The exact opposite would be true for the function $p \mapsto P(E)u(p-1) + (1-P(E))u(p)$. However, unless u is the identity function on the real line, the “neutrality” of the assessment of subjective probability would be compromised.

Thus, for the individual, P(E) is the maximum price at which he/she would be willing to buy a “simple bet based on E”, as well as the minimum price at which he/she would be willing to sell such a bet.

For a seller of “a simple bet based on E” (in a thought experiment) the probability of the event P(E) is the “(actuarily) fair premium” that the seller would charge for fully insuring against the loss of 1 unit of money if event E occurs. Hence, for a seller of a “simple bet based on E” in the above thought experiment, P(E) could be the “relative frequency” of event E. I would be inclined to view this as something like “price equal to average cost”, where cost includes “normal profits”. P(E) is the minimum price that the seller would expect for a “simple bet based on E”.

Due to “asymmetry of information” the subjective probability of an event may be different for different individuals- in particular, between buyers and sellers of insurance policies.

According to the “Ramsey-de Finetti interpretation of mathematical (Kolmogrov) conditional probability”, if the “perceived” probability of the event F is positive, the “probability of (the event) E conditional on (the event) F” that is perceived by (seller) buyer of a “simple bet based on F” and a “simple bet based on $E \cap F$” (in a thought experiment) is the “perceived share of” the price of a “simple bet based” on F, that the (seller) buyer would price a “simple bet based on $E \cap F$” for.

Thus, if $P(F) > 0$, then $P(E|F)$ is the unique value of p that satisfies $pP(F)[P(E\cap F)-1] + (1-pP(F))P(E\cap F) = 0$.

It is easy to see that if X is the sample space and F = X, so that P(X) = 1, then $P(E|X) = p = P(E)$.

The buyers of simple bets or more generally lottery tickets, use probabilities in the sense of prices of simple bets to evaluate their use value or expected utility. The evaluation procedure is explained in the fourth section of [6]. This procedure is consistent with what we would expect from the probability of an event being equal to the price of a simple bet based on it, namely, given two events E and F with their respective probabilities for the buyer satisfying Q(E) > Q(F), and two lottery tickets- the first yielding a monetary gain if E occurs and nothing otherwise, and the second yielding exactly the same monetary gain if F occurs and nothing otherwise, the buyer will prefer the first lottery ticket to the second if the monetary gain is positive and the buyer will prefer the second lottery ticket to the first if the monetary gain is negative.

3. St. Petersburg Paradox

We now provide a critique as well as a resolution of the well-known St. Petersburg paradox. In what follows we quote extensively from the discussion on the paradox that is available in [6].

The paradox that we are concerned with results from a "thought experiment" consisting "of repeated trials of an unbiased coin till the first head shows up. If the first head shows up on the nth toss, then the participant in this game gets $\$2^n$". How much should a person be willing to pay to play this game?

While no reasonable person would be willing to pay more than a couple of dollars for it, apparently an expected monetary value maximiser should be willing to pay an unbounded sum of money to play the game, provided one believes the coin is fair and not loaded in favor of showing either heads or tails. However, the conclusion that an expected monetary value maximiser would be willing to stake any amount of money that is conceivable, to participate in such a game, rests crucially on the assumption that simply because ⋯ the coin under consideration is fair and unbiased" and the outcome on each toss is independent of the outcome on any other" the probability-as perceived by the expected monetary value maximise- of the first head occurring on the n^{th} toss is $(\frac{1}{2})^n$. Our contention is that this paradox results, not because the willingness to pay for participating in such an experiment is based on expected monetary value, but because the individual would assign probability $(\frac{1}{2})^n$ to the first head occurring on the n^{th} toss. There is an underlying assumption the there is a total and complete suspension of judgement by the individual leading him/her to the interpret the statement "independent tosses of a fair coin" exactly the way it is understood in Kolmogorov (mathematical) probability theory.

Unlike the approach adopted above, subjective probabilities would allow an individual who is offered the opportunity to participate in such a game to assign the conditional probability of a head on the n^{th} toss given tails on the previous "n-1" tosses, to be equal to $\frac{1}{2}$ for n = 1, 2, 3 and be equal to 1 for n = 4. With these conditional probabilities and assuming that the probability of a head being realized on the first toss is $\frac{1}{2}$, the expected monetary value of the experiment under question would be \$5 (i.e., $3 + \frac{1}{8} \times 1 \times 16$).

The conditional probabilities that we have suggested to explain the St. Petersburg paradox, is one among innumerable possibilities, that not only contests the assumption that human understanding of the uncertainty inherent in a "thought experiment" or "demonstration" involving an infinite sequence of coin tosses that are fair and whose outcome on any toss is independent of the outcome on any other, is programmed to process the information provided the way it is done in probability theory, but goes a step further and challenges the next best assumption, that it may not mimic the behaviour of a robot that is programmed to interpret the inherent or underlying uncertainty, even as a Markov process. The St. Petersburg paradox, as such, does not seem to provide a reason for expected utility theory.

What may be an argument against using expected monetary value as a reasonable evaluator of uncertain prospects, is the assertion that some individuals may consider \$5 too high a price for participating in the experiment. Using expected utility with a Bernoulli utility function for gains and losses u such that there exists a positive integer w > 0 for which $u(n) = (n + w)^{\frac{1}{2}} - w^{\frac{1}{2}}$ for all integers $n \geqslant -w$ and for $x \in [n-1,n]$, $u(x) = u(n-1) + [u(n) - u(n-1)](x-n+1)$ for all for all integers $n \geqslant -w + 1$, the expected utility from participating in such an experiment would be $\frac{1}{2}(2 + w)^{\frac{1}{2}} + \frac{1}{4}(4 + w)^{\frac{1}{2}} + \frac{1}{8}(8 + w)^{\frac{1}{2}} + \frac{1}{8}(16 + w)^{\frac{1}{2}} - w^{\frac{1}{2}}$, since $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$. For example (and for the sake of simplicity) suppose w = 1. Then the expected utility is $\frac{1}{2}(3)^{\frac{1}{2}} + \frac{1}{4}(5)^{\frac{1}{2}} + \frac{1}{8}(9)^{\frac{1}{2}} + \frac{1}{8}(17)^{\frac{1}{2}} - 1$ which is approximately equal to 1.315. If u(x) is equal to 1.315 then x is greater than 4 but strictly less than 5. Thus, using expected utility instead of expected monetary value to evaluate uncertain prospects may lead to the individual willing to pay less than \$5 to participate in the experiment.

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Conflicts of Interest

The author declares no conflict of interest.

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